Phys 366 Mathematical Methods of Physics Fall 2016

Homework Assignment #6 Due Friday, October 7
(20 points) (at lecture)

6.1 (10 points) **Orthonormal bases.** Suppose $|v_1\rangle, \ldots, |v_d\rangle$ are linearly independent (not necessarily normalized) vectors in a d-dimensional complex vector space; these vectors span the space. Define the normalized vector $|e_1\rangle = |v_1\rangle / ||\,|v_1\rangle\,||$, and for $j = 2, \ldots, d$, define normalized vectors $|e_j\rangle$ inductively by

$$|e_j\rangle = \frac{(I - P_{j-1})|v_j\rangle}{||\,(I - P_{j-1})|v_j\rangle\,||}\;, \qquad P_{j-1} = \sum_{k=1}^{j-1} |e_j\rangle\langle e_j|\;.$$

The operator $I - P_{j-1}$ projects *orthogonal* to the subspace spanned by the first $j-1$ vectors, so this procedure proceeds by taking each vector $|v_j\rangle$, and projecting it orthogonal to the previous vectors $|e_k\rangle$, $k = 1, \ldots, d-1$, and normalizing the result. This procedure is called *Gram-Schmidt orthogonalization.* It should yield an orthonormal basis.

(a) *Show* that the vectors $|e_j\rangle$, $j = 1, \ldots, d$, are an orthonormal basis.

Now suppose that the vectors $|e_j\rangle$, $j = 1, \ldots, d$, are vectors, not *a priori* normalized or orthogonal, in a d-dimensional complex vector space and that they satisfy the completeness relation

$$\sum_{j=1}^{d} |e_j\rangle\langle e_j| = I\;.$$

(b) *Show* that these vectors, just because of the completeness property, make up an orthonormal basis. (Hint: This is entirely an exercise in being clever. First show that these vectors span the space, then argue that they are linearly independent, and then use the linear independence plus the completeness relation to show that they are orthonormal.)

6.2 (10 points) Challenge problem