Phys 366 Mathematical Methods of Physics Fall 2016

Homework Assignment #6 Due Friday, October 7
(20 points) (at lecture)

6.2 (10 points) Challenge problem. Consider a two-dimensional complex vector space. Let $|+,z\rangle$ and $|-,z\rangle$ be an orthonormal basis in this space. These two kets are eigenvectors, with eigenvalues ± 1, of the Hermitian operator

$$\sigma_z = |+,z\rangle\langle +,z| - |-,z\rangle\langle -,z| \; ,$$

i.e., $\sigma_z|+,z\rangle = |+,z\rangle$ and $\sigma_z|-,z\rangle = -|-,z\rangle$ (be sure you know why this is true). The notation $|\pm,z\rangle$ for the eigenvectors should be read as "the eigenvector of σ_z with eigenvalue ± 1.

The states of a spin-$\frac{1}{2}$ particle are described in a two-dimensional complex vector space. The z component of the particle's angular momentum is the observable $S_z = \frac{1}{2}\hbar\sigma_z$, so the state $|+,z\rangle$ describes the particle having spin up in the z direction, with eigenvalue $\frac{1}{2}\hbar$, and the state $|-,z\rangle$ describes the particle having spin down in the z direction, with eigenvalue $-\frac{1}{2}\hbar$.

The matrix that represents σ_z in the $|\pm,z\rangle$ basis is

$$\begin{pmatrix} \langle +,z|\sigma_z|+,z\rangle & \langle +,z|\sigma_z|-,z\rangle \\ \langle -,z|\sigma_z|+,z\rangle & \langle -,z|\sigma_z|-,z\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \longleftrightarrow \quad \sigma_z \; .$$

Here and throughout this problem, we adopt the convention that first rows and first columns are labeled by the eigenvalue $+1$ and second rows and second columns are labeled by the eigenvalue -1. *You need now to sear into your consciousness how to read this notation, because if you don't work at it, it looks pretty much like gobbledygook.*

(a) Consider now the two kets

$$|+,y\rangle = \frac{1}{\sqrt{2}}(|+,z\rangle + i|-,z\rangle) \; , \qquad |-,y\rangle = \frac{1}{\sqrt{2}}(|+,z\rangle - i|-,z\rangle) \; .$$

Give the column-vector representations of these two vectors in the $|\pm,z\rangle$ basis. *Write* the expressions and the row-vector representations for the corresponding bras. *Show* that the y kets make up an orthonormal basis. By inverting your expressions, *write* the z kets in terms of the y kets and the z bras in terms of the y bras.

(b) Consider now the operator

$$\sigma_y = -i|+,z\rangle\langle -,z| + i|-,z\rangle\langle +,z| \; .$$

Give the matrix representation of σ_y in the $|\pm,z\rangle$ basis. *Show* that the kets $|\pm,y\rangle$ are eigenvectors of σ_y with eigenvalues ± 1.

The notation $|\pm,y\rangle$ for the eigenvectors should be read as "the eigenvector of σ_y with eigenvalue ± 1. The y component of the angular momentum of a spin-$\frac{1}{2}$ particle is the observable $S_y = \frac{1}{2}\hbar\sigma_y$, so the state $|+,y\rangle$ describes the particle having spin up in the y direction, with eigenvalue $\frac{1}{2}\hbar$, and the state $|-,y\rangle$ describes the particle having spin down in the y direction, with eigenvalue $-\frac{1}{2}\hbar$.

(c) *Give* the bra-ket outer-product forms of σ_z and σ_y in the $|\pm,y\rangle$ basis. *Give* the matrix representations of σ_z and σ_y in the $|\pm,y\rangle$ basis.

An arbitrary ket can be written in either basis, with a corresponding column-vector representation:

$$
\begin{aligned}
|\psi\rangle &= c_+|+,z\rangle + c_-|-,z\rangle \quad \longleftrightarrow \quad \begin{pmatrix} c_+ \\ c_- \end{pmatrix} , \\
|\psi\rangle &= d_+|+,y\rangle + d_-|-,y\rangle \quad \longleftrightarrow \quad \begin{pmatrix} d_+ \\ d_- \end{pmatrix} .
\end{aligned}
$$

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The corresponding expressions for bras and their row-vector representation are the following:

$$
\begin{aligned}
\langle\psi| &= \langle +,z|c_+^* + \langle -,z|c_-^* \quad \longleftrightarrow \quad \begin{pmatrix} c_+^* & c_-^* \end{pmatrix} , \\
\langle\psi| &= \langle +,y|d_+^* + \langle -,y|d_-^* \quad \longleftrightarrow \quad \begin{pmatrix} d_+^* & d_-^* \end{pmatrix} .
\end{aligned}
$$

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(d) The (passive) transformation between the coefficients $c_\pm$ and the coefficients $d_\pm$ is given by

$$
d_\xi = \sum_\eta V_{\xi\eta} c_\eta \ , \tag{1}
$$

where η and ξ are indices that take on the values $\pm$ and $V = ||V_{\xi\eta}||$ is a 2×2 unitary matrix. Beginning with the transformation (1), *write* this transformation in three equivalent ways obtained by inverting (1) and taking a complex conjugate, and then *write* the four equivalent transformations as matrix equations involving column and row vectors and V or $V^\dagger$.

(e) Let U be the unitary operator (active transformation) that maps the z basis to the y basis:

$$
|\pm,y\rangle = U|\pm,z\rangle \ .
$$

Find the matrix elements of U in the z basis, and *relate* them to the matrix elements V.