Phys 366 Mathematical Methods of Physics Fall 2016

Homework Assignment #7 Due Friday, October 21
(30 points) (at lecture)

7.1 (10 points) Consider a two-dimensional complex vector space. Let $|+,z\rangle$ and $|-,z\rangle$ be an orthonormal basis. These two states are eigenstates, with eigenvalues ± 1, of the Hermitian operator

$$\sigma_z = |+,z\rangle\langle +,z| - |-,z\rangle\langle -,z| \quad \longleftrightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \tag{1}$$

i.e., $\sigma_z|+,z\rangle = |+,z\rangle$ and $\sigma_z|-,z\rangle = -|-,z\rangle$. The notation $|\pm,z\rangle$ for the eigenstates should thus be read as "the eigenstate of σ_z with eigenvalue ± 1.

The states of a spin-$\frac{1}{2}$ particle are described in a two-dimensional complex vector space. The z component of the particle's angular momentum is the observable $S_z = \frac{1}{2}\hbar\sigma_z$, so the state $|+,z\rangle$ describes the particle having spin up in the z direction, with eigenvalue $\frac{1}{2}\hbar$, and the state $|-,z\rangle$ describes the particle having spin down in the z direction, with eigenvalue $-\frac{1}{2}\hbar$.

The matrix on the right in Eq. (1) is the representation of σ_z in the $|\pm,z\rangle$ basis, with the convention that the first row and first column of the matrix are labeled by the eigenvalue $+1$ and the second row and second column are labeled by the eigenvalue -1. Writing this out explicitly, we have

$$\begin{pmatrix} \langle +,z|\sigma_z|+,z\rangle & \langle +,z|\sigma_z|-,z\rangle \\ \langle -,z|\sigma_z|+,z\rangle & \langle -,z|\sigma_z|-,z\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

You need to continue searing into your consciousness how to read this notation, because if you don't work at it, it looks pretty much like gobbledygook.

Consider now the Hermitian operators

$$\sigma_x = |+,z\rangle\langle -,z| + |-,z\rangle\langle +,z| \quad \longleftrightarrow \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

$$\sigma_y = -i|+,z\rangle\langle -,z| + i|-,z\rangle\langle +,z| \quad \longleftrightarrow \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

The three operators σ_x, σ_y, and σ_z are called the *Pauli operators*, and their matrix representations in the z basis are called the Pauli matrices. The operators $S_x = \frac{1}{2}\hbar\sigma_x$ and $S_y = \frac{1}{2}\hbar\sigma_y$ are the x and y components of a spin-$\frac{1}{2}$ particle's angular momentum.

(a) By calculating the nine operator products, *show* that the Pauli operators satisfy

$$\sigma_j\sigma_k = \delta_{jk}I + i\epsilon_{jkl}\sigma_l ,$$

where the final term uses the summation convention. You can calculate the products either using the bra-ket notation or using the matrix representations in the z basis.

The result of part (a) shows that the Pauli operators square to the identity, i.e., $\sigma_j^2 = I$ for $j = x, y, z$, and anti-commute, i.e., $\sigma_j\sigma_k = -\sigma_k\sigma_j$ for $j \neq k$, and that their commutators are

$$[\sigma_j, \sigma_k] = \sigma_j\sigma_k - \sigma_k\sigma_j = \delta_{jk}I + i\epsilon_{jkl}\sigma_l - \delta_{kj}I - i\epsilon_{kjl}\sigma_l = 2i\epsilon_{jkl}\sigma_l \ .$$

These commutation relations mean that the components of angular momentum satisfy the *canonical commutation relations* for angular momentum, i.e.,

$$[S_j, S_k] = (\tfrac{1}{2}\hbar)^2[\sigma_j, \sigma_k] = (\tfrac{1}{2}\hbar)^2 2i\epsilon_{jkl}\sigma_l = i\hbar\epsilon_{jkl}S_l \ .$$

(b) *Find* the eigenvalues and eigenvectors of σ_x and σ_y (normalize the eigenvectors). You should find that the eigenvalues are ± 1, and you should denote the eigenvectors by $|\pm, x\rangle$ and $|\pm, y\rangle$. *Verify* that both pairs of eigenvectors make up an orthonormal basis. *Write* out σ_x in its eigenbasis and σ_y in its eigenbasis.

(c) Suppose we have a quantum system in the state $|+, z\rangle$. What are the probabilities for finding the results ± 1 in a measurement of the observable σ_z? σ_x? σ_y?

(d) Suppose we have a quantum system in the state $|+, x\rangle$. What are the probabilities for finding the results ± 1 in a measurement of the observable σ_z? σ_x? σ_y?

(e) Suppose we have a quantum system in the state $|+, y\rangle$. What are the probabilities for finding the results ± 1 in a measurement of the observable σ_z? σ_x? σ_y?

(f) Suppose you are handed a quantum system that is prepared in the state $|+, z\rangle$ with probability 1/2 and in the state $|-, x\rangle$ with probability 1/2. What are the probabilities for finding the results ± 1 in a measurement of the observable σ_z? σ_x? σ_y?

(g) *Show* that all the probabilities of part (e) are matrix elements of the operator

$$\rho = \frac{1}{2}|+, z\rangle\langle +, z| + \frac{1}{2}|-, x\rangle\langle -, x| \ .$$

This operator is called the *density operator* for the state prepared by the procedure in part (e). *Write* out ρ and its matrix representation in the $|\pm, z\rangle$ basis. Then find the eigenvalues and eigenvectors of ρ.

If you do this entire problem successfully, you will understand how to do a good fraction of quantum mechanics, even though you didn't expect to learn anything about quantum mechanics in this course.

7.2 (10 points) The state of a spin-1 particle is described in a three-dimensional complex vector space. We let $|m, z\rangle$, where $m = +1, 0, -1$, be an orthonormal basis for this space. These three states are eigenstates, with eigenvalues $+\hbar$, $0\hbar$, and $-\hbar$, of the Hermitian operator

$$J_z = \sum_m m\hbar|m, z\rangle\langle m, z| = \hbar\big(|1, z\rangle\langle 1, z| - |-1, z\rangle\langle -1, z|\big) \quad \longleftrightarrow \quad \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ ; \quad (1)$$

i.e., $J_z|1,z\rangle = \hbar|1,z\rangle$, $J_z|0,z\rangle = 0$, and $J_z|-1,z\rangle = -\hbar|-1,z\rangle$. The notation $|m,z\rangle$ for the eigenstates should thus be read as "the eigenstate of J_z with eigenvalue $m\hbar$.

The matrix on the right in Eq. (1) is the representation of J_z in the $|m,z\rangle$ basis, with the convention that the first row and first column of the matrix are labeled by the eigenvalue $+\hbar$, the second row and second column are labeled by the eigenvalue $0\hbar$, and the third row and third column are labeled by the eigenvalue $-\hbar$. Writing this out explicitly, we have

$$\begin{pmatrix} \langle 1,z|\sigma_z|1,z\rangle & \langle 1,z|\sigma_z|0,z\rangle & \langle 1,z|\sigma_z|-1,z\rangle \\ \langle 0,z|\sigma_z|1,z\rangle & \langle 0,z|\sigma_z|0,z\rangle & \langle 0,z|\sigma_z|-1,z\rangle \\ \langle -1,z|\sigma_z|1,z\rangle & \langle -1,z|\sigma_z|0,z\rangle & \langle -1,z|\sigma_z|-1,z\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

The x and y components of the particle's angular momentum are the Hermitian operators

$$\begin{aligned} J_x &= \frac{\hbar}{\sqrt{2}}\Big(|1,z\rangle\langle 0,z| + |0,z\rangle\langle 1,z| + |0,z\rangle\langle -1,z| + |-1,z\rangle\langle 0,z|\Big) \\ &\longleftrightarrow \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \\ J_y &= -\frac{i\hbar}{\sqrt{2}}\Big(|1,z\rangle\langle 0,z| - |0,z\rangle\langle 1,z| + |0,z\rangle\langle -1,z| - |-1,z\rangle\langle 0,z|\Big) \\ &\longleftrightarrow -\frac{i\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} . \end{aligned}$$

(a) *Calculate* the nine operator products J_jJ_k, $j,k = x,y,z$, and use these to *show* that $J_x^2 + J_y^2 + J_z^2 = 2\hbar^2 I$ and that the angular-momentum components obey the commutation relations $[J_j, J_k] = i\hbar\epsilon_{jkl}J_l$ (here we let $x = 1$, $y = 2$, and $z = 3$, as we often do in ordinary three-dimensional space). These commutation relations are called the *canonical commutation relations* for angular momentum. (Hint: These calculations are tedious at best. The point is that they are a less tedious if you do the operator multiplications using the bra-ket notation and the orthonormality of the states $|m,z\rangle$, instead of doing matrix multiplication.)

(b) *Find* the eigenvalues and eigenvectors of J_x and J_y. You should find that both these operators have the same eigenvalues as J_z. Label the eigenvectors of J_x by $|m,x\rangle$ and the eigenvectors of J_y by $|m,y\rangle$, where $m = +1, 0, -1$.

(c) Suppose the particle is in the state $|\psi\rangle = |0,x\rangle = (|1,z\rangle - |-1,z\rangle)/\sqrt{2}$. What are the probabilities of finding the results $m\hbar$ in a measurement of J_z? J_x? J_y?

7.3 (10 points) Challenge problem.