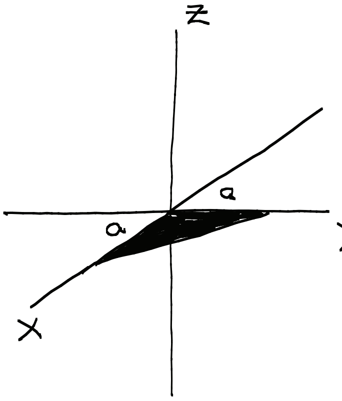
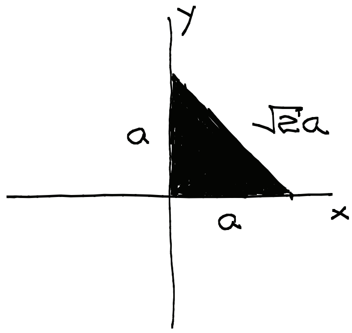
Phys 366 Mathematical Methods of Physics Fall 2016

Homework Assignment #8 **Due Thursday, November 3**
(40 points) **(at lecture)**

8.1 (10 points) Consider a homogeneous isosceles right triangle with edge lengths a (hypotenuse $\sqrt{2}a$) and surface mass density σ. The total mass of the triangle is $M = \frac{1}{2}\sigma a^2$.





(a) *Find* the inertia tensor relative to the origin with the axes and Cartesian coödinates shown in the drawing.

(b) The principal axes and the principal moments of inertia are the eigenvectors and eigenvalues of the inertia tensor. *Determine* the principal axes and principal moments by diagonalizing the inertia tensor found in part (a).

Switch now to using the principal axes and the associated Cartesian coördinates. The change in the inertia tensor between the original coördinates and the new coördinates is an example of how the inertia tensor changes under an orthogonal transformation, i.e., a rotation, of the coördinate system.

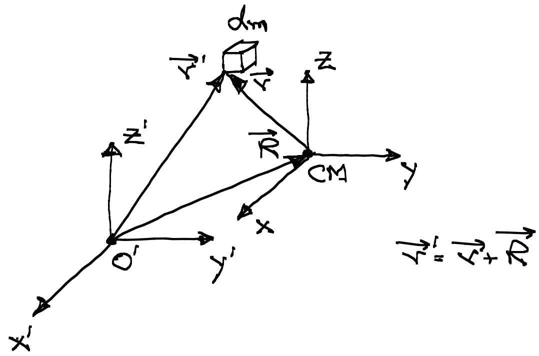
(c) *Calculate* the inertia tensor directly with respect to the original origin and the principal axes found in part (b), thus verifying the principal axes and principal moments found in part (b).

(d) The center of mass (CM),

$$\mathbf{R} = \frac{1}{M}\int dm\,\mathbf{r}\ ,$$

where M is the total mass, is the mass-weighted average position within a mass distribution. *Find* the CM of the triangle.

(e) Suppose we have two Cartesian coördinate systems with parallel axes, one with origin at the CM and the other with origin at O', as shown in the drawing.



Let $\mathbf{R}$ be the vector from O' to the CM. *Show* that the inertia tensor relative to O' is related to the inertia tensor relative to the CM by

$$I'_{jk} = M(R^2\delta_{jk} - X_jX_k) + I^{(\mathrm{CM})}_{jk} \ .$$

This is called the *parallel-axis theorem.* You should read it as saying that the inertia tensor relative to O' is the sum of (i) an inertia tensor relative to O' as though all the mass were concentrated at the CM and (ii) the inertia tensor relative to the center of mass.

(f) For the triangle, use the parallel-axis theorem to *find* the inertia tensor with respect to the CM with axes parallel to the principal axes of parts (b) and (c).

8.2 (10 points) **Energy-momentum conservation for the vacuum electromagnetic field.** The Maxwell equations for the electromagnetic field in vacuum are (cgs Gaussian units; c is the speed of light)

$$\begin{aligned}
\nabla\cdot\mathbf{E} &= 0 \ ,\\
\nabla\cdot\mathbf{B} &= 0 \ ,\\
\nabla\times\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \ ,\\
\nabla\times\mathbf{B} &= \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \ .
\end{aligned}$$

(a) *Show* that

$$\frac{\partial u}{\partial t} = -\nabla\cdot\mathbf{S} \ , \tag{1}$$

where

$$u = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) \qquad \text{and} \qquad \mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} \ .$$

Equation (1) has the classic form of a conservation law for a scalar quantity, so one interprets u as the energy density density in the electromagnetic field and $\mathbf{S}$, which is called the *Poynting vector*, as the flux of electromagnetic energy. This interpretation has stood the test of time. The conservation law of Eq. (1) is, as usual, more easily interpreted in its integral form:

$$\frac{d}{dt}\int_V u\,d\tau = -\int_V \nabla\cdot\mathbf{S}\,d\tau = -\oint_S \mathbf{S}\cdot d\mathbf{a} \ .$$

Suppose now that we have a little box of length L and cross-sectional area A moving along at the speed of light in the direction of the length L. It takes a time $T = L/c$ for the box to pass by. The energy in the box is $E = uAL$, and the flux of energy is $E/AT = cE/AL = cu = S$. Relativity teaches us that for motion at the speed of light, the linear momentum is related to the energy by $P = E/c$, so the momentum density is $P/AL = E/cAL = S/c^2$. From this little argument, we are motivated to think of

$$\frac{\mathbf{S}}{c^2} = \frac{1}{4\pi c}\mathbf{E}\times\mathbf{B}$$

as the density of linear momentum in the electromagnetic field.

(b) *Show* that

$$\frac{1}{c^2}\frac{\partial \mathbf{S}}{\partial t} = -\nabla\cdot\mathbf{T}\ , \tag{2}$$

where

$$\mathbf{T} = \frac{1}{4\pi}\Big(-\mathbf{E}\otimes\mathbf{E} - \mathbf{B}\otimes\mathbf{B} + \frac{1}{2}\mathbf{I}(E^2+B^2)\Big)\ .$$

is the *Maxwell stress tensor*. In component form, the Maxwell stress tensor looks like

$$T_{jk} = \frac{1}{4\pi}\Big(-E_jE_k - B_jB_k + \frac{1}{2}\delta_{jk}(\mathbf{E}^2+\mathbf{B}^2)\Big)\ .$$

and the divergence of a 2-tensor is a vector defined by

$$\nabla\cdot\mathbf{T} = \hat{\mathbf{e}}_j T_{jk,k}\ .$$

Equation (2) has the classic form of a conservation law for a vector quantity. Again, the integral is clearer:

$$\frac{d}{dt}\int_V \frac{\mathbf{S}}{c^2}\,d\tau = -\int_V \nabla\cdot\mathbf{T}\,d\tau = -\oint_S \mathbf{T}\cdot d\mathbf{a}\ .$$

The integral of $\mathbf{S}/c^2$ over V gives the electromagnetic linear momentum in V. What the equation says is that the electromagnetic momentum in V changes because electromagnetic momentum flows in through the bounding surface S.

The conservation laws (1) and (2) can be generalized to include the source of electromagnetic fields, i.e., moving charged particles. These are incorporated by including charge and current density as sources in the Maxwell equations. What changes in the conservation laws is that both conservation laws acquire a source term: the energy conservation law (1) has a source term for the work that the electromagnetic field does on charged particles, and the momentum conservation law (2) has a source term for the change in mechanical momentum of the charged particles due to the electromagnetic (Lorentz) force on the particles. These generalized conservation laws, incorporating both mechanical and electromagnetic energy and momentum, were one of the triumphs of 19th Century physics, providing strong evidence that energy and momentum are robust concepts that lie at the heart of the way the world works.

8.3 (10 points) Boas Sec. 10.3, Problems 3, 4, 5, 6, 7

8.4 (10 points) Challenge problem