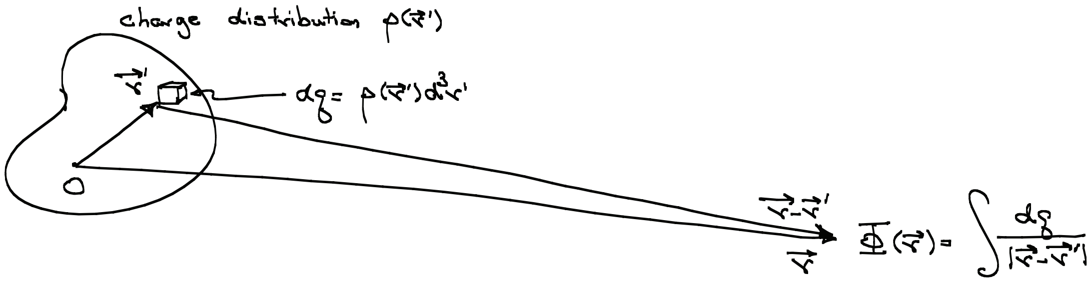
Phys 366 Mathematical Methods of Physics Fall 2016

Homework Assignment #8 Due Thursday, November 3
(40 points) (at lecture)

8.4 (10 points) Challenge problem. **Multipole moments.** The electrostatic potential of a distribution of charge with charge density $\rho(\mathbf{r})$ is (cgs Gaussian units)

$$\Phi(\mathbf{r}) = \int d^3r' \, \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \; .$$

We are interested in the potential far away from a localized source, as shown in the drawing; thus our assumption is that for all points in the source, $r' = |\mathbf{r}'| \ll |\mathbf{r}| = r$.



(a) By expanding to second order in powers of r'/r, *show* that

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r}\left[1 + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{1}{2}\left(-\frac{r'^2}{r^2} + \frac{3(\mathbf{r}\cdot\mathbf{r}')^2}{r^4}\right) + \ldots\right] ,$$

where the three dots stand for terms that are of order $(r'/r)^3$ or smaller.

(b) *Show* that to second order in powers of r'/r, the electrostatic potential is given by

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{Q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + \frac{1}{2}\sum_{j,k}\frac{Q_{jk}x_j x_k}{r^5} + \ldots \\ &= \frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} + \frac{1}{2}\frac{\hat{\mathbf{r}}\cdot\mathbf{Q}\cdot\hat{\mathbf{r}}}{r^3} + \ldots ,\end{aligned}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector pointing in the direction of $\mathbf{r}$,

$$Q = \int d^3r \, \rho(\mathbf{r})$$

is the total charge,

$$\mathbf{p} = \int d^3r \, \rho(\mathbf{r})\mathbf{r}$$

is the *dipole-moment* vector, and

$$Q_{jk} = \int d^3r \, \rho(\mathbf{r})(3x_j x_k - r^2\delta_{jk})$$

are the components of the *quadrupole-moment* tensor $\mathbf{Q}$. The quadrupole-moment tensor is a rank-two tensor; notice that it is symmetric and has zero trace. The official way to write it abstractly is

$$\mathbf{Q} = \int d^3r \, \rho(\mathbf{r})(3\,\mathbf{r} \otimes \mathbf{r} - r^2\mathbf{I}) \ ,$$

where $\mathbf{I}$ is the unit tensor; using Dirac notation, one could also write it as

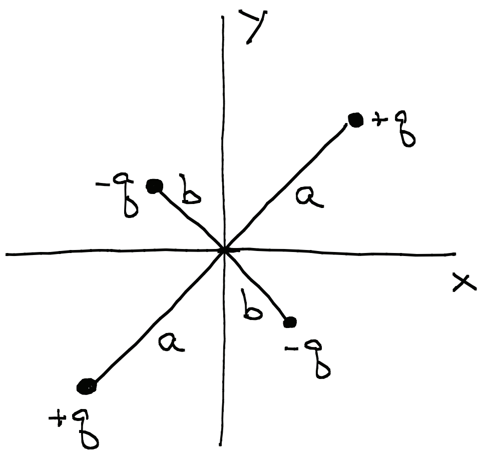
$$\mathbf{Q} = \int d^3r \, \rho(\mathbf{r})(3\,|\mathbf{r}\rangle\langle\mathbf{r}| - r^2\mathbf{I}) \ ,$$

where $\mathbf{I}$ is the unit operator.

(c) *Show* that the electric field $\mathbf{E} = -\nabla\Phi$ to the same order is given by

$$\mathbf{E} = \frac{Q\,\hat{\mathbf{r}}}{r^2} + \frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3} + \frac{1}{2}\frac{5(\hat{\mathbf{r}}\cdot\mathbf{Q}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - 2\mathbf{Q}\cdot\hat{\mathbf{r}}}{r^4} + \dots \ .$$

Consider the charge distribution shown below. Charges $\pm q$ are located at 45° angles to the x and y axes, with the distances from the origin shown. It is easy to see that the total charge and the dipole moment of this distribution are zero.



(d) *Find* the components of the quadrupole-moment tensor.

(e) *Find* the eigenvalues (principal components) and eigenvectors (principal directions) of the quadrupole-moment tensor. (Hint: You might want to guess the eigenvectors.)

(f) *Calculate* directly the quadrupole-moment tensor in the Cartesian coördinate system associated with the eigenbasis, and thereby *verify* that it is diagonal and has principal components equal to the eigenvalues.