Phys 366 Mathematical Methods of Physics Fall 2016

Homework Assignment #9 Due Thursday, November 17
(50 points) (at lecture)

9.1 (10 points) Boas Sec. 7.5, Problems 4, 5, 6, and 11

9.2 (10 points) Consider a function $f(x)$ equal to a periodic function $f_L(x)$ within the interval $[-L/2, L/2]$, but is set to zero outside this interval. We can summarize the situation by

$$f(x) = \begin{cases} f_L(x)\ , & \text{for } |x| \le L/2, \\ 0\ , & \text{for } |x| > L/2. \end{cases}$$

Show how to construct the Fourier transform $\tilde{f}(k)$ from the Fourier-series coefficients

$$c_n = \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} dx\, f(x) e^{-ik_n x}$$

for the periodic function $f_L(x)$. *Show* that $\tilde{f}(k_n) = \sqrt{L} c_n$.

9.3 (10 points) **Nyquist sampling theorem.** An arbitrary function of time can be represented by

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \tilde{f}(\omega) e^{-i\omega t}\ ,$$

where the Fourier transform $\tilde{f}(\omega)$ is given by

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} dt\, f(t) e^{i\omega t}\ .$$

Suppose now that the function $f(t)$ is *band-limited.* This means that the Fourier transform has frequencies no higher than some frequency Ω—i.e., $f(\omega) = 0$ for $|\omega| \ge \Omega$—so $f(t)$ can be written as

$$f(t) = \int_{-\Omega}^{\Omega} \frac{d\omega}{2\pi}\, \tilde{f}(\omega) e^{-i\omega t}\ .$$

Show how to construct $f(t)$ from the function values $f(t_j)$, where the times t_j are evenly spaced a time π/Ω apart, i.e., $t_j = j\pi/\Omega$, $j = -\infty, \ldots, \infty$. This fundamental construction, called the *Nyquist sampling theorem*, shows that to reconstruct *exactly* a band-limited function, you only need to sample its values at times spaced by a half-period of the highest frequency in the function. This is the way the digital information on a CD reproduces music (mp3 files are digital files that are further compressed). [Hint: This might seem like a problem completely out of the blue, which you have no clue how to do, but you should think about the previous problem.]

9.4 (10 points) **Fourier series and Fourier transforms.** Consider a periodic function $f_L(x)$, i.e., one that satisfies $f_L(x+L) = f(x)$; it can be expanded as a Fourier series,

$$f_L(x) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} c_n e^{ik_n x} \,, \qquad c_n = \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} dx\, f_L(x) e^{-ik_n x} \,, \qquad k_n = \frac{2\pi n}{L} \,.$$

Show that the Fourier transform of $f_L(x)$ is

$$\tilde{f}_L(k) = \frac{2\pi}{\sqrt{L}} \sum_{n=-\infty}^{\infty} c_n \delta(k - k_n) \,.$$

9.5 (10 points) Challenge problem.