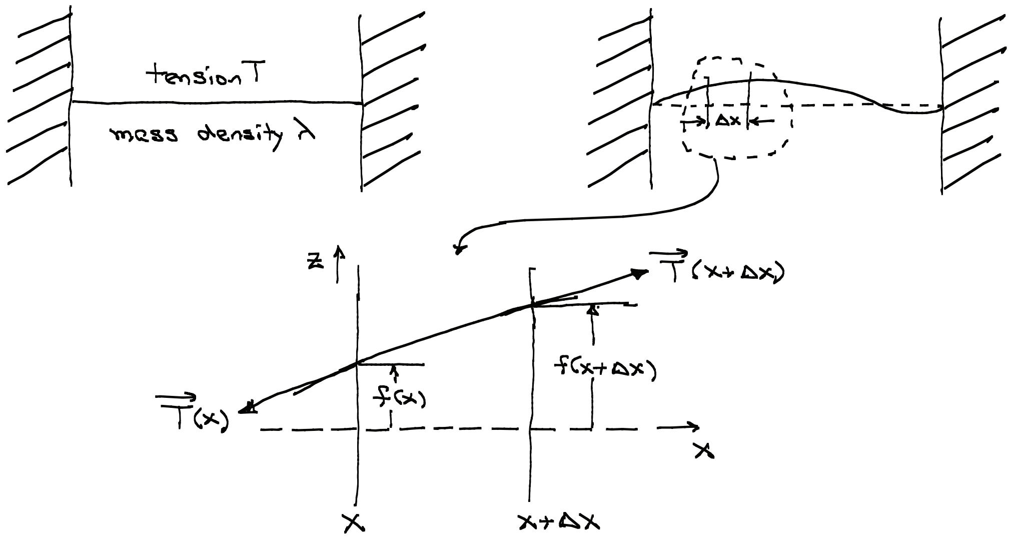
Phys 366 Mathematical Methods of Physics Fall 2016

Homework Assignment #9 Due Thursday, November 17
(50 points) (at lecture)

9.5 (10 points) Challenge problem. Consider a string that has uniform linear mass density λ. the string is stretched between two walls till it is under tension T. The string is shown below: on the left, the string is in its equilibrium configuration, where it is a straight line between the walls; on the right, the string is plucked so that its vertical displacement from equilibrium is $z = f(x,t)$. The inset shows the displacement and forces on a small segment of the string of length Δx. We are going to study the behavior of the string when its vertical displacement is small, i.e., $|\partial f/\partial x| \ll 1$.

For a string under even moderate tension, the magnitude of the tension, T, stays constant as the string undergoes small oscillations; in addition, gravity can be neglected since it is small compared to the tension forces.



(a) *Show* that the force on the small segment is

$$F = T\left(\left.\frac{\partial f}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial f}{\partial x}\right|_{x}\right) = T\Delta x\frac{\partial^2 f}{\partial x^2}\ .$$

Using Newton's second law, *show* that the time-dependent displacement $f(x,t)$ obeys the equation

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0\ , \tag{1}$$

where $v = \sqrt{T/\lambda}$.

Equation (1) is called the one-dimensional *wave equation*. It, its three-dimensional counterpart,

$$\nabla^2 f - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}\ ,$$

and various generalizations of these govern wave phenomena and thus are among the most important equations in physics. The remainder of the problem explores ways to solve the one-dimensional wave equation.

(b) *Transform* the wave equation to coördinates $\xi = x - vt$ and $\eta = x + vt$; these coördinates are called *characteristic coördinates. Solve* the resulting equation and thereby *show* that the general solution of the one-dimensional wave equation is a superposition of a wave of arbitrary shape moving to the right and another wave of arbitrary shape moving to the left.

Now we are going to solve the equation using Fourier transforms. For that purpose, we will consider the walls to be so far away that we can forget about them—or that we are dealing with a phenomenon that doesn't require walls—and thus that the wave equation applies for all x.

(c) In this part, we Fourier transform with respect to x, but not t:

$$f(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\tilde{f}(k,t)e^{ikx}\;, \qquad \tilde{f}(k,t) = \int_{-\infty}^{\infty} dx\, f(x,t)e^{-ikx}\;.$$

Find the equation satisfied by $\tilde{f}(k,t)$, *solve* for the general solution, and *show* that when transformed back to $f(x,t)$, it agrees with your result for part (b).

(d) In this part, we Fourier transform with respect to both x and t:

$$f(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\,\tilde{f}(k,\omega)e^{i(kx-\omega t)}\;, \quad \tilde{f}(k,\omega) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt\, f(x,t)e^{-i(kx-\omega t)}\;.$$

Find the equation satisfied by $\tilde{f}(k,\omega)$, *solve* for the general solution, and *show* that when transformed back to $f(x,t)$, it agrees with your result for parts (b) and (c).