10.1 **Pascal's triangle**

We need to show that

$$\binom{N}{n} = \binom{N-1}{n-1} + \binom{N-1}{n} .$$

We first need to prove this for the two endpoints where $n = 0$ and $n = N$; since $\binom{N}{0} = \binom{N}{N} = 1$, the formula is trivial for the endpoints, provided we declare that $\binom{N}{-1} = \binom{N}{N+1} = 0$:

$$1 = \binom{N}{0} = \binom{N-1}{-1} + \binom{N-1}{0} = 1 , \qquad 1 = \binom{N}{N} = \binom{N-1}{N-1} + \binom{N-1}{N} = 1 .$$

Now, when $0 < n < N$, we have:

$$\begin{aligned}
\binom{N-1}{n-1} + \binom{N-1}{n} &= \frac{(N-1)!}{(n-1)!(N-n)!} + \frac{(N-1)!}{n!(N-n-1)!} \\
&= \frac{(N-1)!}{(n-1)!(N-n-1)!} \underbrace{\left(\frac{1}{N-n} + \frac{1}{n} \right)}_{= \frac{N}{n(N-n)}} \\
&= \frac{N!}{n!(N-n)!} \\
&= \binom{N}{n} .
\end{aligned}$$