10.2 **Distinguishable particles, bosons, and fermions**

(a) Each distinguishable particle can be in one of L single-particle states, so the total number of ways of assigning states to particles is $D_{N,L} = L^N$.

(b) The overall state of the N fermions is given by a list, $n_1, n_2, \ldots, n_L$, where $n_j = 0, 1$ is the number of particles in state j. This list is a bit sequence of length L, with a 0 saying there is no particle in the state and a 1 saying there is a particle in the state. Any bit sequence of length L, with N zeroes, is an allowed state. The number of such sequence is $\binom{L}{N} = \dfrac{L!}{N!(L-N)!}$. Notice that there is no way at all to fit in all the fermions if $N > L$, so

$$F_{N,L} = \begin{cases} \dbinom{L}{N} = \dfrac{L!}{N!(L-N)!}\,, & N = 0, 1, \ldots, L, \\ 0\,, & N > L. \end{cases}$$

The one state when there no fermions, i.e., $N = 0$ and $F_{0,L} = 1$, is called the vacuum state.

(c) The overall state of the N fermions is given by a list, $n_1, n_2, \ldots, n_L$, where n_j is the number of particles in state j. It is instructive to write this list as a bit sequence, where the numbers n_j are translated to an equivalent number of zeroes and different n_j are separated by a 1:

$$\underbrace{00\ldots0}_{n_1\text{ 0s}}1\underbrace{00\ldots0}_{n_2\text{ 0s}}1 \quad \ldots \quad 1\underbrace{00\ldots0}_{n_L\text{ 0s}}\ .$$

These sequences have length $N + L - 1$ and N 0s and $L - 1$ 1s; it should be clear that every such sequence specifies a state, so the total number of allowed states is

$$B_{N,L} = \binom{N+L-1}{N} = \frac{(N+L-1)!}{N!(L-1)!}\ .$$

Notice that when there are no bosons, i.e., $N = 0$, there is only one state, and that is the vacuum state.

(d) We have the following exact formulas:

$$\begin{aligned} F_{N,L} &= \frac{L(L-1)\cdots(L-N+2)(L-N+1)}{N!} = \frac{L^N}{N!}\left(1-\frac{1}{L}\right)\left(1-\frac{2}{L}\right)\cdots\left(1-\frac{N-1}{L}\right), \\ B_{N,L} &= \frac{(L+N-1)(L+N-2)\cdots(L+1)L}{N!} = \frac{L^N}{N!}\left(1+\frac{N-1}{L}\right)\left(1+\frac{N-2}{L}\right)\cdots\left(1+\frac{1}{L}\right). \end{aligned}$$

When $N \ll L$, all the factors in big parentheses limit to 1, so we have

$$F_{N,L} \simeq B_{N,L} \simeq \frac{L^N}{N!} = \frac{D_{N,L}}{N!}$$

When there are far more states than particles, fermions and bosons act the same way, because even for bosons, there is rarely more than one particle per state. Distinguishable particles will also almost always have just one particle per state; the difference between distinguishable and indistinguishable particles is that the $N!$ permutations of each configuration count as different overall states for distinguisable particles, but they don't count as different states for the indistinguishable particles, whether fermions or bosons. When $N \ll L$, you will often see a distinguishable counting used, with a factor of $1/N!$ added to convert to the result for indistinguishable particles.