11.1

(a) Since $\partial/\partial x$ becomes ik in the Fourier transform, $\tilde{f}(k,t)$ satisfies the ordinary differential equation

$$\frac{d\tilde{f}(k,t)}{dt} = -\alpha^2 k^2 \tilde{f}(k,t) \ .$$

The general solution of this equation with initial value $\tilde{f}(k,0)$ at $t=0$ is

$$\tilde{f}(k,t) = \tilde{f}(k,0)e^{-\alpha^2 k^2 t} \ .$$

(b) Translating back to the spatial domain means to do the following:

$$f(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{f}(k,t)e^{ikx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \tilde{f}(k,0)e^{-\alpha^2 k^2 t}e^{ikx} \ .$$

If we want to get an answer in terms of the initial value $f(x,0)$, we should plug in the Fourier transform of $\tilde{f}(k,0)$ (what we are doing is deriving the convolution property; you could use it directly, but it is a nuisance to remember it):

$$\begin{aligned} f(x,t) &= \int_{-\infty}^{\infty} dx'\, f(x',0) \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-\alpha^2 k^2 t}e^{ik(x-x')}}_{=\frac{e^{-(x-x')^2/4\alpha^2 t}}{\sqrt{4\pi\alpha^2 t}} = G(x-x',t)} \\ &= \int_{-\infty}^{\infty} dx'\, f(x',0)G(x-x',t) \ . \end{aligned}$$

Here we use the Gaussian integral

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2}e^{bx} = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a} \ , .$$

The Green function $G(x-x',t)$ is a normalized Gaussian; it describes the diffusion away from an initial δ-spike at position x'.

We often write the Green function as $G(x,t;x',0)$ to emphasize that it depends on two sets of variables: two initial-condition or source variables, a position x' and a time, here $t'=0$, and two final or field variables, a position x and a time t. Here our notation emphasizes that this Green function depends only on the differences $x-x'$ and $t-t'=t$.