11.2

(a) If we retain only the constant and linear terms in the dispersion relation, we get

$$f(x,t) = e^{i(k_0 x - \omega_0 t)} \underbrace{\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{F}(k) e^{ik(x - v_g t)}}_{= F(x - v_g t)} .$$

This is a carrier wave at wave number k_0, which moves to the right at the phase velocity $v_p = \omega_0/k_0$ within an envelope $F(x - v_g t)$. The envelope $F(x - v_g t)$ which moves to the right at the group velocity v_g. Notice that $F(x)$ gives the shape of the envelope and is equal to the envelope at $t = 0$.

You can see the phenomenon of a carrier wave moving at a different speed than the envelope in any dispersive wave. Water waves are a good example. The Schrödinger equation for a free, nonrelativistic particle is another good example: since $\omega = \hbar k^2/2m$, the phase velocity, $\omega/k = \hbar k/2m = p/2m$, is half the velocity of a classical particle, but the group velocity $v_g = d\omega/dk = \hbar k/m = p/m$, is the same as the classical velocity.

(b) If we also retain the quadratic term in the dispersion relation, the wave packet takes the form

$$f(x,t) = e^{i(k_0 x - \omega_0 t)} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{F}(k) e^{ik(x - v_g t)} e^{-i\alpha k^2 t/2} .$$

The envelope is given by $F(x) = e^{-x^2/2\sigma^2}/\sqrt{2\pi\sigma^2}$, and its Fourier transform is

$$\tilde{F}(k) = \int_{-\infty}^{\infty} dx\, F(x) e^{-ikx} = \int_{-\infty}^{\infty} dx\, e^{-x^2/2\sigma^2} e^{-ikx} .$$

We can either look up this Fourier transform or use the Gaussian integral

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2} e^{bx} = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a} ,$$

which requires that a have a positive real part. This gives

$$\tilde{F}(k) = e^{-\sigma^2 k^2/2} .$$

Plugging this into the expression for the wave packet, we get

$$\begin{aligned}
f(x,t) &= e^{i(k_0 x - \omega_0 t)} \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x - v_g t)} e^{-(\sigma^2 + i\alpha t)k^2/2}}_{= \sqrt{\frac{2\pi}{\sigma^2 + i\alpha t}} \exp\left(-\frac{(x - v_g t)^2}{2(\sigma^2 + i\alpha t)}\right)} \\
&= e^{i(k_0 x - \omega_0 t)} \underbrace{\frac{1}{\sqrt{2\pi(\sigma^2 + i\alpha t)}}}_{= \sqrt{\frac{1}{2\pi\sqrt{\sigma^4 + \alpha^2 t^2}}}\, e^{-i\phi(t)}} \underbrace{\exp\left(-\frac{(x - v_g t)^2}{2(\sigma^2 + i\alpha t)}\right)}_{= \exp\left(-\frac{(x - v_g t)^2(\sigma^2 - i\alpha t)}{2(\sigma^4 + \alpha^2 t^2)}\right)} \\
&= e^{i[k_0 x - \omega_0 t - \phi(t)]} \sqrt{\frac{1}{2\pi\sqrt{\sigma^4 + \alpha^2 t^2}}} \exp\left(-\frac{(x - v_g t)^2}{2(\sigma^2 + \alpha^2 t^2/\sigma^2)}\right) \exp\left(i\alpha t \frac{(x - v_g t)^2}{2(\sigma^4 + \alpha^2 t^2)}\right) .
\end{aligned}$$

Here we use the Gaussian integral again, and we use the following two forms of the complex denominator:

$$\frac{1}{\sigma^2 + i\alpha t} = \frac{\sigma^2 - i\alpha t}{\sigma^4 + \alpha^2 t^2} = \frac{1}{\sqrt{\sigma^4 + \alpha^2 t^2}}\, e^{-2i\phi(t)} , \qquad \tan 2\phi(t) = \frac{\alpha t}{\sigma^2} .$$

The phase shift $\phi(t)$ is a tiny shift in space and time of the carrier wave, the final position-dependent phase puts an envelope-scale oscillation on the envelope function, and the big square root changes the overall amplitude of the wave in a way consistent with the spreading. The key term is the real Gaussian: it describes a wave packet moving to the right with the group velocity and spreading (dispersing) to something that extends over a few times $\sqrt{\sigma^2 + \alpha^2 t^2/\sigma^2}$; the narrower the initial wave packet, the faster the spreading.

An advanced topic one can get from this is that in the limit $\sigma^2 \to 0$, the initial envelope becomes a δ-function. This violates our assumptions about the initial wave packet not being too narrow, but our approximate equations don't know anything about the assumptions we made for their validity. The point is that if we take the limit $\sigma^2 \to 0$ on the wave packet at time t,

$$f(x,t) \underset{\sigma^2 \to 0}{\longrightarrow} e^{i(k_0 x - \omega_0 t)} \frac{1}{\sqrt{2\pi i \alpha t}} \exp\left(i \frac{(x - v_g t)^2}{2\alpha t} \right) = G(x,t) \ ,$$

the result is the wave packet that comes from an initial δ-function envelope at $x = 0$. Since there is nothing special about $x = 0$, the wave packet that comes from an initial δ-function envelope at position x' is $G(x - x', t)$. We can then write the wave packet that comes from *any* initial envelope $F(x)$ as

$$f(x,t) = \int_{-\infty}^{\infty} dx' \, G(x - x', t) F(x') \ .$$

$G(x - x', t)$ is the *temporal Green function*, sometimes called the propagator, for this sort of dispersion. We often write the Green function as $G(x, t; x', 0)$ to emphasize that it depends on two sets of variables: two initial-condition or source variables, a position x' and a time, here $t' = 0$, and two final or field variables, a position x and a time t. Here our notation emphasizes that this Green function depends only on the differences $x - x'$ and $t - t' = t$.

To see this formally, we can manipulate the general expression for the wave packet as follows:

$$\begin{aligned}
f(x,t) &= e^{i(k_0 x - \omega_0 t)} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, \tilde{F}(k) e^{ik(x - v_g t)} e^{-i\alpha k^2 t/2} \\
&= \lim_{\sigma^2 \to 0} e^{i(k_0 x - \omega_0 t)} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, \tilde{F}(k) e^{-\sigma^2 k^2/2} \sigma^2 e^{ik(x - v_g t)} e^{-i\alpha k^2 t/2} \\
&= \int_{-\infty}^{\infty} dx' \, F(x') \underbrace{\lim_{\sigma^2 \to 0} e^{i(k_0 x - \omega_0 t)} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, e^{-\sigma^2 k^2/2} \sigma^2 e^{ik(x - x' - v_g t)} e^{-i\alpha k^2 t/2}}_{= G(x - x', t)} \\
&= \int_{-\infty}^{\infty} dx' \, G(x - x', t) F(x') \ .
\end{aligned}$$