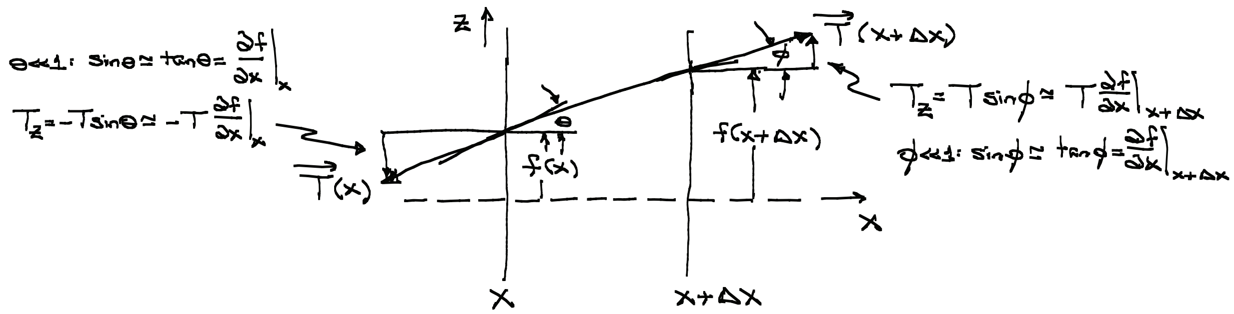
9.5



(a) The drawing shows that the z (vertical) component of the force on the left end of the segment of length Δx is $T_z = -T(\partial f/\partial x)|_x$ and the z component of the force on the right end of the segment is $T_z = T(\partial f/\partial x)|_{x+\Delta x}$. Thus the total force on the segment is

$$F = T\left(\left.\frac{\partial f}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial f}{\partial x}\right|_{x}\right) = T\Delta x\frac{\partial^2 f}{\partial x^2}\ ,$$

where the second form follows from the definition of the second derivative, assuming Δx is infinitesimal.

The mass of the segment is $\lambda\Delta x$, and its acceleration is $\partial^2 f/\partial t^2$, so Newton's law says that

$$T\Delta x\frac{\partial^2 f}{\partial x^2} = F = \lambda\Delta x\frac{\partial^2 f}{\partial t^2}\ .$$

The result is the one-dimensional wave equation,

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0\ , \qquad v^2 = \frac{T}{\lambda}\ . \tag{1}$$

(b) We have to transform the derivatives to the new coördinates $\xi = x - vt$ and $\eta = x + vt$. The efficient way to do that is to regard the partial derivatives as operators and to derive how to go from one set of operators to the other:

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\ ,\\ \frac{\partial}{\partial t} &= \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = -v\frac{\partial}{\partial \xi} + v\frac{\partial}{\partial \eta}\ .\end{aligned}$$

Now, taking this to second derivatives, we have

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \underbrace{\frac{\partial^2}{\partial \eta\,\partial \xi} + \frac{\partial^2}{\partial \xi\,\partial \eta}}_{=2\frac{\partial^2}{\partial \eta\,\partial \xi}}\ ,\\ \frac{\partial^2}{\partial t^2} &= \left(-v\frac{\partial}{\partial \xi} + v\frac{\partial}{\partial \eta}\right)\left(-v\frac{\partial}{\partial \xi} + v\frac{\partial}{\partial \eta}\right) = v^2\frac{\partial^2}{\partial \xi^2} + v^2\frac{\partial^2}{\partial \eta^2} - v^2\underbrace{\left(\frac{\partial^2}{\partial \eta\,\partial \xi} + \frac{\partial^2}{\partial \xi\,\partial \eta}\right)}_{=2\frac{\partial^2}{\partial \eta\,\partial \xi}}\ .\end{aligned}$$

Thus the wave equation transforms to

$$0 = \frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 4\frac{\partial^2 f}{\partial\eta\,\partial\xi}\ . \tag{2}$$

We can solve Eq. (2) one derivative at a time: $\partial f/\partial\eta$ has zero derivative with respect to ξ, $\partial f/\partial\eta$ is equal to an arbitrary function of η, which is a constant as far as ξ is concerned; we write this arbitrary function as $f_L'(\eta)$, so we have

$$\frac{\partial f}{\partial\eta} = f_L'(\eta)\ ;$$

integrating this, we get

$$f(\xi,\eta) = f_L(\eta) + f_R(\xi)\ ,$$

where $f_R(\xi)$ is an arbitrary function of ξ, which is a constant as far as η is concerned. So the general solution is a sum of an arbitrary function of ξ and an arbitrary function of η. Translated to x and t, this means that the general solution is a sum of an wave of arbitrary shape propagating to the right and a wave of arbitrary shape propagating to the left:

$$f(x,t) = f_R(x-vt) + f_L(x+vt)\ ,$$

(c) Recalling that a spatial derivative is equivalent to multiplying the Fourier transform by ik, we get that the equation satisfied by $\tilde{f}(k,t)$ is the

$$\frac{d^2\tilde{f}(k,t)}{dt^2} + (vk)^2\tilde{f}(k,t) = 0\ .$$

This is an ordinary differential equation because the wave number k is just a constant parameter as far as the equation is concerned. The general solution should be familiar:

$$\tilde{f}(k,t) = \tilde{f}_R(k)e^{-ikvt} + \tilde{f}_L(k)e^{ikvt}\ .$$

Here $\tilde{f}_R(k)$ and $\tilde{f}_L(k)$ are arbitrary functions of the wave number k. We get the solution of the wave equation by doing the inverse Fourier transform to find

$$f(x,t) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\tilde{f}(k,t)e^{ikx} = \underbrace{\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\tilde{f}_R(k)e^{ik(x-vt)}}_{=\,f_R(x-vt)} + \underbrace{\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\tilde{f}_L(k)e^{ik(x+vt)}}_{=\,f_L(k+vt)}\ .$$

We see that the general solution is a superposition of a wave traveling to the right and a wave traveling to the left:

$$f(x,t) = f_R(x-vt) + f_L(x+vt)\ .$$

Moreover, we see that the waves traveling to the right and to the left are completely arbitrary and that they are built up from arbitrary superpositions of monochromatic waves.

(d) A spatial derivative is equivalent to multiplying the Fourier transform by ik, and a temporal derivative is equivalent to multiplying the Fourier transform by $-i\omega$, so the Fourier transform satisfies the equation

$$\left(-k^2 + \frac{\omega^2}{v^2}\right)\tilde{f}(k,\omega)\ .$$

Evidently, we must have $\tilde{f}(k,\omega) = 0$ unless $\omega = \pm vk$. Fortunately, we can use the δ-function to write the general solution as

$$\tilde{f}(k,\omega) = 2\pi\tilde{f}_R(k)\delta(\omega - vk) + 2\pi\tilde{f}_L(k)\delta(\omega + vk)\ .$$

[The factors of 2π are chosen with forethought so that our ultimate solution looks just like the results of parts (b) and (c), but sufficient experience with the δ-functions of wave numbers and frequencies would encourage this choice without any forethought.] Plugging this into the Fourier transform, we have

$$\begin{aligned}
f(x,t) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \tilde{f}(k,\omega) e^{i(kx-\omega t)} \\
&= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{f}_R(k) e^{ikx} \underbrace{\int_{-\infty}^{\infty} d\omega\, \delta(\omega - vk) e^{-i\omega t}}_{=e^{-ikvt}} + \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{f}_L(k) e^{ikx} \underbrace{\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \delta(\omega + vk) e^{-i\omega t}}_{=e^{ikvt}} \\
&= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{f}_R(k) e^{ik(x-vt)} + \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{f}_L(k) e^{ik(x+vt)} \\
&= f_R(x-vt) + f_L(x+vt)\ .
\end{aligned}$$