Partial differential equations. II
Physics 366
Lecture 25

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Consider the diffusion equation with a source $s(x,t)$:

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{\alpha^2}\frac{\partial f}{\partial t} = -s\ .$$

We're going to show how to get the general (causal) solution of the diffusion equation with source from the Green function for the sourceless diffusion equation. This is a standard technique for getting the Green function for a differential equation with a source, called an *inhomogeneous equation*, from the initial-value solution of the corresponding equation without source, called a *homogeneous equation*.

For this purpose, let's consider a source of strength s_0 that is confined to a tiny length interval Δx centered at position x_0 and that turns on only for a tiny time interval from time $t_0 - \Delta t$ to time t_0, i.e.,

$$s(x',t') = \begin{cases} s_0\ , & \text{for } x_0 - \Delta x/2 < x' < x_0 + \Delta x/2 \text{ and } t_0 - \Delta t < t' < t_0, \\ 0\ , & \text{otherwise.} \end{cases}$$

Just below, we're going to use the integral form of the diffusion equation with source, applied to the spacetime interval $\Delta x \Delta t$ to show that at time t_0,

$$f(x',t_0) = \begin{cases} \alpha^2 s_0 \Delta t\ , & \text{for } x_0 - \Delta x/2 < x' < x_0 + \Delta x/2, \\ 0\ , & \text{otherwise,} \end{cases} \tag{1}$$

but before getting to this, let's see why we want to show it. The reason is that Eq. (1) is our key result, key in that it is always the way we get started in going from a Green function for a homogeneous problem to a Green function for the corresponding inhomogeneous problem. The reason this works is that this $f(x',t)$ serves as an initial condition just after the short interval when the source is on; we can thus use the initial-value solution of problem 11.1 (now with initial time t_0) to write, for $t > t_0$,

$$f(x,t) = \int_{-\infty}^{\infty} dx'\, G(x-x', t-t_0) f(x',t_0) = G(x-x_0, t-t_0)\alpha^2 s_0 \Delta x \Delta t\ ,$$

where

$$G(x-x_0, t-t_0) = \frac{e^{-(x-x_0)^2/4\alpha^2(t-t_0)}}{\sqrt{4\pi\alpha^2(t-t_0)}}$$

is the Green function without source from problem 11.1. We often write the Green function as $G(x,t;x_0,t_0)$ to emphasize how it depends on two sets of variables: two initial-condition or source variables, a position x_0 and a time t_0, and two final or field variables, a position x and a time t. Here our notation emphasizes that this Green function depends only on the differences $x - x_0$ and $t - t_0$.

To summarize, provided we can show Eq. (1), we now have the solution for the initial condition produced by a source of strength s_0 at position x_0 within Δx and at time t_0 within Δt. This solution applies for $t > t_0$, and $f(x,t)$ vanishes for $t < t_0$. We can put in a step function to enforce this causal condition:

$$f(x,t) = H(t-t_0)G(x-x_0, t-t_0)\alpha^2 s_0 \Delta x \Delta t\ . \tag{2}$$

We could avoid the step function by saying the Green function vanishes for $t < t_0$, but the step function reminds us to make this happen.

Now back to showing Eq. (1). We begin with the integral form of the diffusion equation with source:

$$\frac{1}{\alpha^2}\frac{d}{dt}\int_{x_1}^{x_2} dx\, f(x,t) = \frac{\partial f(x,t)}{\partial x}\bigg|_{x=x_2} - \frac{\partial f(x,t)}{\partial x}\bigg|_{x=x_1} + \int_{x_1}^{x_2} dx\, s(x,t)\ .$$

For x' in the tiny interval Δx and at the time $t_0 - \Delta t$, the integral form of the diffusion equation becomes

$$\frac{1}{\alpha^2}\Delta x \frac{\partial f(x',t')}{\partial t'}\bigg|_{t'=t_0-\Delta t} = \frac{\partial f(x',t_0-\Delta t)}{\partial x'}\bigg|_{x'=x_0-\Delta x/2} - \frac{\partial f(x',t_0-\Delta t)}{\partial x}\bigg|_{x'=x_0+\Delta x/2} + s_0\Delta x\ .$$

The spatial derivatives vanish because they are evaluated at time $t_0 - \Delta t$, before the source turns on. We can approximate the temporal derivative as

$$\frac{\partial f(x',t')}{\partial t'}\bigg|_{t'=t_0-\Delta t} = \frac{f(x',t_0) - f(x',t_0-\Delta t)}{\Delta t} = \frac{f(x',t_0)}{\Delta t}\ ,$$

where $f(x', t_0 - \Delta t) = 0$ because f vanishes before the source turns on. Putting all this together, we have

$$f(x',t_0) = \alpha^2 s_0 \Delta t\ .$$

Since there are no sources at other positions, we get Eq. (1).

Now we're ready to get the solution for an arbitrary source in terms of an integral over the Green function. Since the diffusion equation is linear, we can add the solutions for all sources to get the general solution, so we simply integrate Eq. (1) over all sources $s(x_0, t_0)\, dx_0\, dt_0$:

$$\begin{aligned} f(x,t) &= \alpha^2 \int_{-\infty}^{\infty} dt_0\, H(t-t_0) \int_{-\infty}^{\infty} dx_0\, G(x-x_0, t-t_0) s(x_0,t_0) \\ &= \alpha^2 \int_{-\infty}^{t} dt_0 \int_{-\infty}^{\infty} dx_0\, G(x-x_0, t-t_0) s(x_0,t_0) \\ &= \alpha^2 \int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} dx'\, G(x-x', t-t') s(x',t')\ . \end{aligned} \tag{3}$$

In the last form, we change the names of the source position and time from (x_0, t_0) to (x', t'). Causality is the fact that the temporal integral stops at time t and can be enforced either by stopping the temporal integral or including the step function.

We're now going to work backward to show directly that Eq. (3) is the solution of the diffusion equation with source $s(x, t)$. We begin by noting that the Green function with the Heaviside function, $H(t-t')G(x-x', t-t')$, is often called the *causal Green function*. The derivation we have done shows that the causal Green function satisfies the equation

$$\frac{\partial^2 H(t-t')G(x-x',t-t')}{\partial x^2} - \frac{1}{\alpha^2}\frac{\partial H(t-t')G(x-x',t-t')}{\partial t} = -\frac{1}{\alpha^2}\delta(x-x')\delta(t-t')\ , \tag{4}$$

i.e., is the solution of the diffusion equation for a δ source that blinks on at time t' at position x', bu we're now going to show directly that the causal Green function satisfies Eq. (4).

For this task, the step function really comes in handy:

$$\begin{aligned} &\frac{\partial^2 H(t-t')G(x-x',t-t')}{\partial x^2} - \frac{1}{\alpha^2}\frac{\partial H(t-t')G(x-x',t-t')}{\partial t} \\ &\quad = -\frac{1}{\alpha^2}\delta(t-t')G(x-x',t-t') + H(t-t')\left(\frac{\partial^2 G(x-x',t-t')}{\partial x^2} - \frac{1}{\alpha^2}\frac{\partial G(x-x',t-t')}{\partial t}\right)\ . \end{aligned}$$

We use the fact that the derivative of the step function is the δ-function. As a normalized Gaussian, the Green function goes to a δ-function when $t \to t'$, so the first term gives the answer we want. For $t < t'$, the last term is zero because of the step function, and for $t > t'$, the Green function satisfies the diffusion equation without source, so the term in big parentheses is zero (we check this directly below). Thus we have

$$\frac{\partial^2 H(t-t')G(x-x',t-t')}{\partial x^2} - \frac{1}{\alpha^2}\frac{\partial H(t-t')G(x-x',t-t')}{\partial t} = -\frac{1}{\alpha^2}\delta(t-t')\delta(x-x') \ , \tag{5}$$

as promised.

Now let's check directly that $G(x-x', t-t')$ satisfies the diffusion equation for $t > t'$:

$$\begin{aligned}
\frac{\partial G(x-x',t-t')}{\partial t} &= -\frac{1}{2(t-t')}G(x-x',t-t') + \frac{(x-x')^2}{4\alpha^2(t-t')^2}G(x-x',t-t') \\
&= \left(-\frac{1}{2(t-t')} + \frac{(x-x')^2}{4\alpha^2(t-t')^2}\right) G(x-x',t-t') \ , \\
\frac{\partial G(x-x',t-t')}{\partial x} &= -\frac{(x-x')}{2\alpha^2(t-t')}G(x-x',t-t') \ , \\
\frac{\partial^2 G(x-x',t-t')}{\partial x^2} &= -\frac{1}{2\alpha^2(t-t')}G(x-x',t-t') + \left(\frac{(x-x')}{2\alpha^2(t-t')}\right)^2 G(x-x',t-t') \\
&= \left(-\frac{1}{2\alpha^2(t-t')} + \frac{(x-x')^2}{4\alpha^4(t-t')^2}\right) G(x-x',t-t') \ , \\
\Longrightarrow \frac{\partial^2 G(x-x',t-t')}{\partial x^2} &- \frac{1}{\alpha^2}\frac{\partial G(x-x',t-t')}{\partial t} = 0 \ .
\end{aligned} \tag{6}$$

We have one last step, to show directly that Eq. (3) is the solution of the inhomogeneous diffusion equation, but this is now easy. Working from Eq. (5), we have

$$\begin{aligned}
\frac{\partial^2 f(x,t)}{\partial x^2} - \frac{1}{\alpha^2}\frac{\partial f}{\partial t} &= \alpha^2 \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dx' \, \underbrace{\left(\frac{\partial^2}{\partial x^2} - \frac{1}{\alpha^2}\frac{\partial}{\partial t}\right) H(t-t')G(x-x',t-t')}_{= -\frac{1}{\alpha^2}\delta(t-t')\delta(x-x')} s(x',t') \\
&= -s(x,t) \ .
\end{aligned}$$

or working from Eq. (6), we have

$$\begin{aligned}
\frac{\partial^2 f(x,t)}{\partial x^2} - \frac{1}{\alpha^2}\frac{\partial f}{\partial t} &= -\int_{-\infty}^{\infty} dx' \, \underbrace{G(x-x',0)}_{=\delta(x-x')} s(x',t) \\
&\quad + \alpha^2 \int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} dx' \, \underbrace{\left(\frac{\partial^2 G(x-x',t-t')}{\partial x^2} - \frac{1}{\alpha^2}\frac{\partial G(x-x',t-t')}{\partial t}\right)}_{=0} s(x',t') \\
&= -s(x,t) \ .
\end{aligned}$$