FOURIER SERIES AND FOURIER TRANSFORMS
Physics 366
Lectures 17–19

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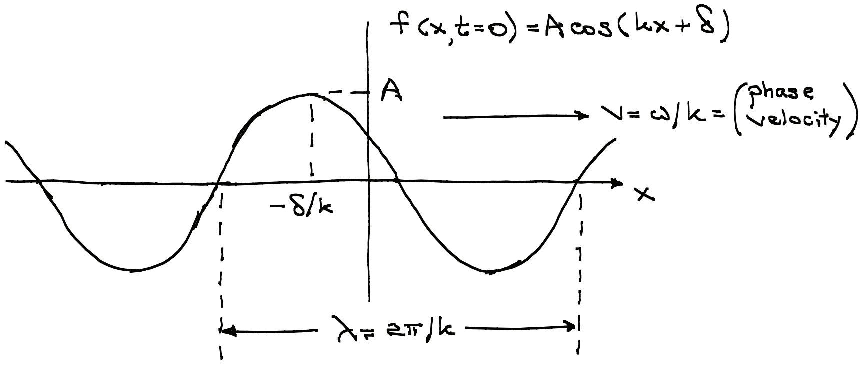
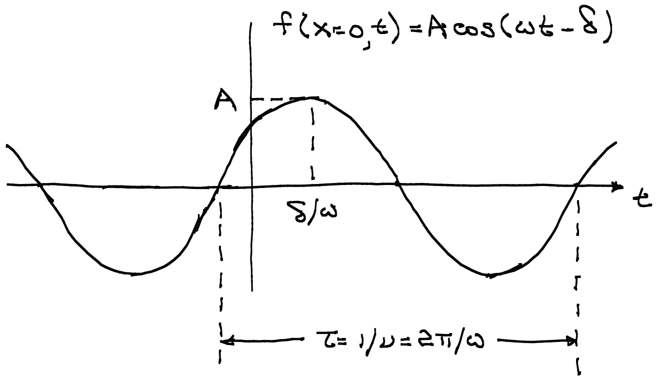
1. INTRODUCTION

We are going to be looking at how to describe and analyze a two-dimensional wave $f(x,t)$—i.e., a function of one spatial variable x and time t. The simplest kind of wave is a monochromatic wave traveling in the $+x$ direction,

$$f(x,t) = A\cos(kx - \omega t + \phi)\ . \tag{1.1}$$

The quantity A is called the (real) *amplitude*, k is the *wave number*, ω is the *angular frequency*, ϕ is the *phase*, and the speed of propagation, called the *phase velocity*, is $\omega/k = v$. The wavelength of the wave is $\lambda = 2\pi/k$, the period is $\tau = 2\pi/\omega$, and the (ordinary) frequency is $\nu = 1/\tau = \omega/2\pi$. Notice that $\lambda\nu = v$.





We will find it useful to use Euler's relation to write $f(x,t)$ as

$$f(x,t) = \mathrm{Re}\Big(Ae^{i\phi}e^{i(kx-\omega t)}\Big)\ . \tag{1.2}$$

The complex quantity $Ae^{i\phi}$ is called the *complex amplitude.* Notice that the imaginary part,

$$\mathrm{Im}\Big(Ae^{i\phi}e^{i(kx-\omega t)}\Big) = A\sin(kx-\omega t+\phi) = A\cos(kx-\omega t+\phi-\pi/2)\ , \tag{1.3}$$

is another wave that is 90° out of phase with the real part.[1]

Since the spatial and temporal dependences of such a wave are related, we don't really need to be considering a function of two variables. We could, for example, sit at one spot, say $x=0$, and examine the wave's temporal dependence, given by $f(x=0,t)$, as it passes that spot. On the other hand, we could study how the wave looks in space at a particular time, say $t=0$; i.e., we could look at the function $f(x,t=0)$. In this document we are generally going to make the latter choice and study the function

$$f(x) \equiv f(x,t=0)\ , \tag{1.4}$$

but you will want to keep in mind that everything we do can also be applied to a function of time. For a monochromatic wave, we have

$$f(x) = A\cos(kx+\phi)\ . \tag{1.5}$$

2. FOURIER SERIES AND SUPERPOSITIONS OF MONOCHROMATIC WAVES

Though a monochromatic wave has the virtue of having a precise angular frequency ω and a precise wave number k, it is unphysical because it extends over all of space. A physically realistic wave, called a *wave packet*, extends only over a finite region of space, with a well-defined start and finish. The purpose of this document is to show how a wave packet can be constructed as a *superposition* of monochromatic waves. Monochromatic waves are important not just because they are simple, but more importantly because they provide the building blocks for all other kinds of waves.

We begin our investigation by considering the superposition of just two monochromatic waves, which have the same amplitude and phase and nearly the same wave number:

$$f(x) = A\cos(k_1x+\phi) + A\cos(k_2x+\phi)\ . \tag{2.1}$$

To see what this function looks like, it is useful to introduce complex amplitudes and to write $f(x)$ as

$$f(x) = \mathrm{Re}\Big(Ae^{i\phi}e^{ik_1x} + Ae^{i\phi}e^{ik_2x}\Big)\ . \tag{2.2}$$

Now we go through some mathematical manipulations,

$$f(x) = \mathrm{Re}\bigg(Ae^{i\phi}e^{i(k_1+k_2)x/2}\Big(e^{i(k_1-k_2)x/2}+e^{-i(k_1-k_2)x/2}\Big)\bigg) = \mathrm{Re}\bigg(Ae^{i\phi}e^{i(k_1+k_2)x/2}\,2\cos\Big(\frac{k_1-k_2}{2}x\Big)\bigg)\ , \tag{2.3}$$

which put $f(x)$ in the form

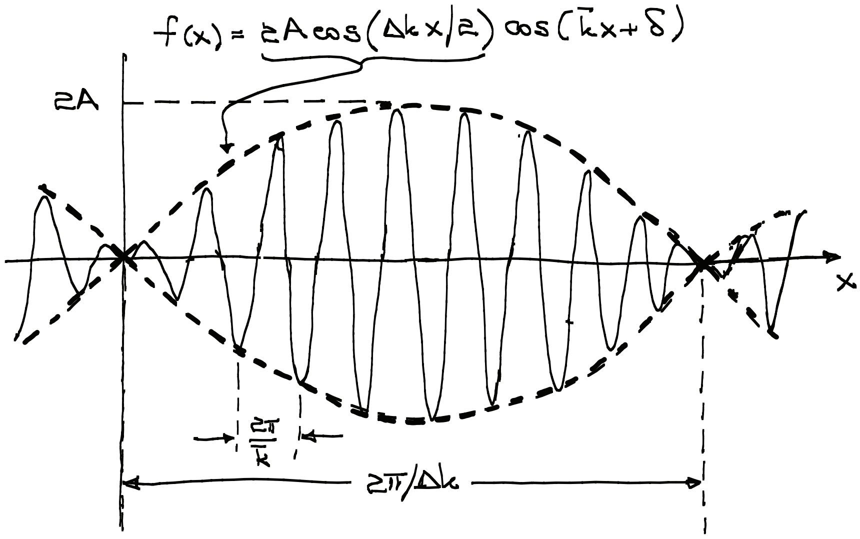
$$f(x) = 2A\cos\Big(\frac{k_1-k_2}{2}x\Big)\cos\Big(\frac{k_1+k_2}{2}x+\phi\Big)\ . \tag{2.4}$$

To say that the two wave numbers are nearly the same means that the wave-number difference $\Delta k \equiv k_1-k_2$ is (in magnitude) much smaller than the average wave number $(k_1+k_2)/2$, i.e.,

$$|\Delta k| \ll \frac{k_1+k_2}{2} \equiv \overline{k}\ .$$

[1] A wave going in the $+x$ direction must have the dependence $kx-\omega t$; the opposite signs between the spatial and temporal parts is what sets the wave going in the $+x$ direction. This is the reason that physicists always use opposite Fourier conventions for space and time. We use e^{ikx} as the Fourier function in space, and $e^{-i\omega t}$ for the Fourier function in time; if you want to convert the following to physicists' temporal convention, you should substitute ωt in place of kx wherever you see a kx. Electrical engineers like to use $e^{i\omega t}$ for temporal Fourier functions, but that's because they didn't look far enough ahead to see the need for describing waves; you can always get from the EE convention to the physics one by doing electrical engineering with $e^{j\omega t}$ and letting $j=-i$.

We can think of $f(x)$ in Eq. (2.4) as a wave $\cos(\overline{k}x + \phi)$ that has the average wave number $\overline{k}$ and the phase ϕ, but whose amplitude $2A\cos(\Delta k\,x/2)$ varies slowly and sinusoidally at half the wave-number difference. When this amplitude has its largest absolute value of $2A$, i.e., when $\Delta k\,x = n\pi$ where n is an *even* integer, the two waves superposed in Eq. (1) are said to interfere *constructively*; in contrast, when the amplitude is zero, i.e., when $\Delta k\,x = n\pi$ where n is an *odd* integer, the waves in Eq. (2.1) are said to interfere *destructively.* The functions $\pm 2A\cos(\Delta k\,x/2)$ form an *envelope* for the rapid oscillations of $f(x)$; the rapid oscillations with wave number $\overline{k}$ are bounded by this envelope. The phenomenon of periodic constructive and destructive interference is often called *beats.*



The lesson here is that in a superposition of two monochromatic waves, destructive interference makes the wave go to zero at certain places. Perhaps if we use more than two monochromatic waves in the superposition, we can arrange the destructive interference so that the function $f(x)$ is nonzero only in an isolated region—*voilà*, a wave packet.

To see how this can be achieved, we will start with what you know about Fourier series: that any periodic function can be written as a superposition of cosines and sines. Specifically, suppose $f_L(x)$ is periodic over a length L. Then we can write $f_L(x)$ as the *Fourier series*

$$f_L(x) = \frac{1}{\sqrt{L}}\left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos k_n x + b_n \sin k_n x\right), \tag{2.5}$$

where the wave numbers

$$k_n = 2n\pi/L \tag{2.6}$$

are chosen so that as n runs from 0 to ∞, we include in the series all the sinusoidal (harmonic) functions (including the constant function) that are periodic over the length L. The reasons for the factor of 1/2 in the constant term and the overall normalization $1/\sqrt{L}$ become clear as we go along (note that Boas does not include the $1/\sqrt{L}$ in the formula for a Fourier series).

We can replace the trigonometric functions $\cos k_n x$ and $\sin k_n x$ by complex exponentials $e^{\pm ik_n x}$ and rewrite the Fourier series (2.5) as

$$f_L(x) = \frac{1}{\sqrt{L}}\sum_{n=-\infty}^{\infty} c_n e^{ik_n x}\,. \tag{2.7}$$

It is sensible to let this sum run over both positive and negative integers n, thereby including both $e^{ik_n x} = e^{i2\pi nx/L}$ and its complex conjugate, $e^{-i2\pi nx/L}$, in the sum without having to have two terms for each positive n, as in Eq. (2.5). The price we pay for this is that we have to remember that n and k_n can be either positive or negative.

By using Euler's relation for the complex exponential, $e^{ik_nx} = \cos k_nx + i \sin k_nx$, we can relate the two kinds of Fourier coefficients:

$$c_{\pm n} = \frac{1}{2}(a_n \mp ib_n) \ , \quad n = 0, 1, \ldots, \infty \ . \tag{2.8}$$

This formula works for $c_0 = \frac{1}{2}a_0$ provided we introduce a mythical $b_0 = 0$. We can invert Eq. (2.8) to give

$$\begin{aligned} a_0 &= 2c_0 \ , & b_0 &= 0 \ , \\ a_n &= c_n + c_{-n} \ , & b_n &= i(c_n - c_{-n}) \ , \quad n = 1, 2, \ldots \ . \end{aligned} \tag{2.9}$$

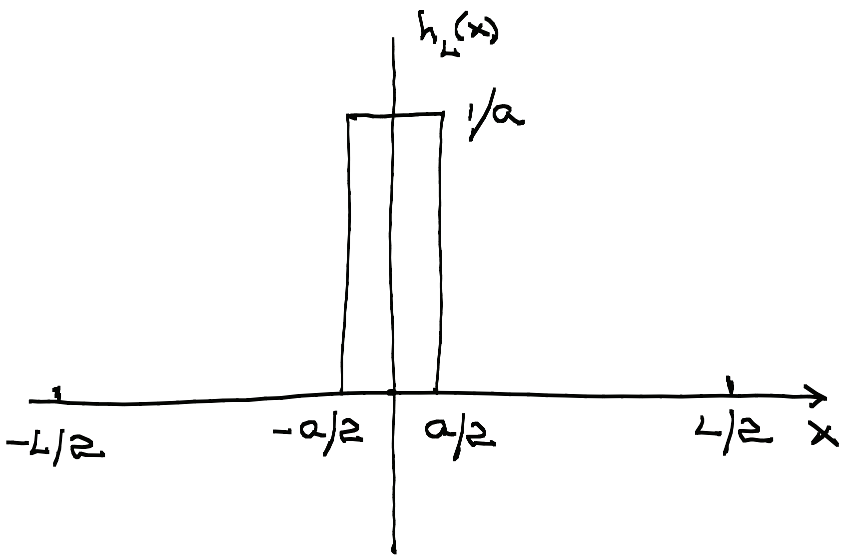
The superposed monochromatic terms in Eq. (2.7) are called the *Fourier components* of the function $f_L(x)$. We can find the *Fourier coefficient* c_n for the nth Fourier component—i.e., we can invert the Fourier series—by doing the integral

$$c_n = \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} dx\, f_L(x) e^{-ik_nx} \ . \tag{2.10}$$

Translated to the coefficients a_n and b_n, the inversion of the Fourier series becomes

$$\left.\begin{aligned} a_n &= \frac{2}{\sqrt{L}} \int_{-L/2}^{L/2} dx\, f_L(x) \cos k_nx \ , \\ b_n &= \frac{2}{\sqrt{L}} \int_{-L/2}^{L/2} dx\, f_L(x) \sin k_nx \ , \end{aligned}\right. \quad n = 0, 1, \ldots, \infty \ . \tag{2.11}$$

To make the $n = 0$ case of these formulas work is the reason we have the factor of $1/2$ in the constant term of Eq. (2.5); notice also that the formula automatically gives $b_0 = 0$, so that case works, too. If $f_L(x)$ is a real function, then a_n and b_n are real, so $c_{-n} = c_n^*$. If $f_L(x)$ is an even function, then $b_n = 0$, so $c_{\pm n} = a_n/2$ for $n = 0, 1, \ldots \infty$; if $f_L(x)$ is an odd function, then $a_n = 0$, so $c_{\pm n} = \mp ib_n/2$, for $n = 0, 1, \ldots \infty$.



Now we're ready to see what kind of Fourier series is required to represent a function that is nonzero only over small part of each period. For specificity, let's consider a function defined on the interval $-L/2 \leq x \leq L/2$ by

$$h_{a,L}(x) = \begin{cases} 1/a \ , & \text{for } |x| \leq a/2, \\ 0 \ , & \text{for } a/2 \leq |x| \leq L/2. \end{cases} \tag{2.12}$$

This function is zero except that it has a flat-topped bump of height $1/a$ and width a (area under the bump always equal to 1) centered at the origin.[2] Of course, since $h_{a,L}(x)$ is periodic, the same bump is repeated at each integral multiple of L. For this function the Fourier coefficients (2.10) become[3]

$$c_n = \frac{1}{\sqrt{L}} \int_{-a/2}^{a/2} dx\, \frac{1}{a} e^{-ik_n x} = \frac{1}{\sqrt{L}} \frac{\sin(k_n a/2)}{k_n a/2} = \frac{1}{\sqrt{L}} \frac{\sin(n\pi a/L)}{n\pi a/L} \ . \tag{2.13}$$

We should record a couple of observations about these Fourier coefficients. First, for small values of $|n|$, for which the period (or wavelength, if you wish) $2\pi/|k_n| = L/|n|$ of the Fourier function is much bigger than a, the Fourier coefficients all have approximately the same value,

$$c_n \simeq \frac{1}{\sqrt{L}} \ , \qquad \text{for } |k_n|a \ll 2\pi. \tag{2.14}$$

The reason is that the Fourier function $e^{ik_n x}$ doesn't have room even to begin to oscillate within the integral (2.13) and thus it can be replaced by its value at the origin, which gives $c_n \simeq c_0$. Second, for large values of $|n|$, for which the period $2\pi/|k_n|$ is much bigger than a, the magnitude of the Fourier coefficients falls off as $1/|n|$. The reason is that in this situation the integral (2.13) averages over many periods of the Fourier function. The lesson is that to represent a function with periodic bumps of width a, separated by distance L, we need to have roughly equal contributions from the Fourier components with $|k_n|a \lesssim 2\pi$, with decreasing contributions from shorter-period Fourier components. The Fourier components interfere constructively within the bumps at each integral multiple of L and interfere destructively otherwise.

A corollary of this lesson occurs in the limit $a \to 0$. The "function" in this limit is called the periodic δ-function:

$$\delta_L(x) \equiv \lim_{a\to 0} h_{a,L}(x) \ . \tag{2.15}$$

The "bump" in $\delta_L(x)$ has zero width, but such enormous height that the integral under it is 1; this is the periodic δ-function, because this bump is repeated at every multiple of L. To represent $\delta_L(x)$ as a Fourier series, we need Fourier components of all orders, all contributing with a coefficient of the same size $c_n = 1/\sqrt{L}$:

$$\delta_L(x) = \frac{1}{L} \sum_{n=-\infty}^{\infty} e^{ik_n x} = \frac{1}{L} \sum_{n=-\infty}^{\infty} e^{i2\pi n x/L} \ . \tag{2.16}$$

The key property of the δ-function is that for any function $f_L(x)$, we have

$$\int_{-L/2}^{L/2} dx\, \delta_L(x) f_L(x) = \lim_{a\to 0} \int_{-L/2}^{L/2} dx\, h_{a,L}(x) f_L(x) = \lim_{a\to 0} \frac{1}{a} \int_{-a/2}^{a/2} dx\, f_L(x) = f_L(0) \ . \tag{2.17}$$

The limit in Eq. (2.15)—and, hence, the function $\delta_L(x)$ itself—only makes sense when one does an integral and takes the limit afterward.

3. FOURIER SERIES AS A COMPLEX VECTOR SPACE

Given two periodic functions $f_L(x)$ and $g_L(x)$, we can define their *inner product* as

$$(g_L, f_L) = \langle g_L | f_L \rangle = \int_{-L/2}^{L/2} dx\, g_L^*(x) f_L(x) \ . \tag{3.1}$$

[2] The function $h_{a,L}(x)$ is analogous to the envelope of the superposition of two monochromatic waves, but unlike that situation,we have not put any rapid oscillations inside the envelope. We could do that, but for our present purposes, it would only add complication without providing additional insight.

[3] The function $\sin x/x$ is sometimes called the *sine cardinal* (don't ask why) function and denoted $\operatorname{sinc} x = \sin x/x$. Some people use a different definition: $\operatorname{sinc} x = \sin(\pi x)/\pi x$. We avoid possible confusion here by not using this abbreviated notation at all.

For the remainder of this section, we will leave off the subscript L, understanding that all of our functions are periodic with period L.

Functions can be added and multiplied by a complex number, suggesting that they can be regarded as elements of a vector space over the complex numbers; the function that takes on the value zero everywhere is the zero of this vector space. The inner product above is the natural inner product on this vector space. The only thing one has to be careful about is that only the functions such that

$$\langle f|f\rangle = \int_{-L/2}^{L/2} dx\, |f(x)|^2 < \infty\ , \tag{3.2}$$

i.e., those for which the integral of the absolute square is finite, are vectors in the vector space. This space is called the vector space of square-integrable functions on the interval $[-L/2, L/2]$. One other thing to be aware of, although it never makes much difference, is that two functions that differ only on a set of measure zero are considered to be the same function, since inner products involving the two functions are the same.

We can now see that the Fourier functions,

$$e_n(x) = \frac{1}{\sqrt{L}}\, e^{ik_n x}\ , \quad n = -\infty, \ldots, \infty, \tag{3.3}$$

are orthonormal under the inner product (3.1):

$$\langle e_m|e_n\rangle = \frac{1}{L}\int_{-L/2}^{L/2} dx\, e^{i(k_n - k_m)x} = \frac{1}{L}\int_{-L/2}^{L/2} dx\, e^{i2\pi(n-m)x/L} = \delta_{nm} \tag{3.4}$$

[the unit normalization here is the reason for the $1/\sqrt{L}$ in the Fourier series (2.5) and (2.7)]. For $n = m$, the result is obvious; for $n \neq m$, the integral covers an integral number of periods of a periodic function, so it is zero. Since any periodic function can be expanded in terms of the Fourier functions, they span the vector space and thus are an *orthonormal basis*. The new feature is that the vector space is infinite-dimensional, and thus this orthonormal basis has an infinite number of vectors in it.

The Fourier series (2.7) and its inverse (2.10) are examples of the expansion of a vector in a complex vector space:

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e_n(x)\ , \qquad & c_n &= \int_{-L/2}^{L/2} dx\, e_n^*(x) f(x)\ , \\ |f\rangle &= \sum_{n=-\infty}^{\infty} c_n |e_n\rangle\ , \qquad & c_n &= \langle e_n|f\rangle\ . \end{aligned} \tag{3.5}$$

Moreover, the inner product can now be written in terms of the Fourier coefficients as

$$\langle g|f\rangle = \sum_{n=-\infty}^{\infty} d_n^* c_n = \sum_{n=-\infty}^{\infty} \langle g|e_n\rangle\langle e_n|f\rangle\ , \tag{3.6}$$

where $d_n = \langle e_n|g\rangle$ is the nth Fourier coefficient for $g(x)$, just like in a finite-dimensional vector space.

The *completeness* of the Fourier functions is the fact that *any* periodic function in the vector space can be expanded in terms of them. This fact can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e_n(x) = \sum_{n=-\infty}^{\infty} e_n(x) \underbrace{\int_{-L/2}^{L/2} dx'\, e_n^*(x') f(x')}_{=\langle e_n|f\rangle} = \int_{-L/2}^{L/2} dx' \left(\sum_{n=-\infty}^{\infty} e_n(x) e_n^*(x')\right) f(x')\ . \tag{3.7}$$

This is simply the statement that if an orthonormal basis is complete, then any vector can be written as a linear combination of the basis vectors, each multiplied by the component along that basis vector—i.e., basis covers all the directions in the vector space. The function in large parentheses is like the unit operator in that when it "operates" on a function, it returns the same function. The reason this works is that

$$\sum_{n=-\infty}^{\infty} e_n(x)e_n^*(x') = \frac{1}{L}\sum_{n=-\infty}^{\infty} e^{ik_n(x-x')} = \frac{1}{L}\sum_{n=-\infty}^{\infty} e^{i2\pi n(x-x')/L} = \delta_L(x-x') \ , \tag{3.8}$$

as one sees from the definition of the periodic δ-function in Eq. (2.16). Equation (3.8) is the *completeness relation* for the Fourier functions.

Advanced Topic. The inner product (3.1) looks just like an ordinary inner product in a complex vector space except that we integrate over a continuous variable x instead of summing over a discrete index n. Thus it is tempting to think in the following way: just as $c_n = \langle e_n|f\rangle$ is the component of the vector $|f\rangle$ obtained by projecting onto the Fourier basis vector $|e_n\rangle$, so $f(x) = \langle x|f\rangle$ is the component of the vector $|f\rangle$ obtained by projecting onto a "position basis vector" $|x\rangle$. To be clear, the way we get the value of the function f at the point x is to project the vector for f onto the position basis vector $|x\rangle$. Just as

$$|f\rangle = \sum_{n=-\infty}^{\infty} |e_n\rangle\langle e_n|f\rangle \ , \tag{3.9}$$

so we should have

$$|f\rangle = \int dx'\, |x'\rangle\langle x'|f\rangle \ , \tag{3.10}$$

but if this is to work, we must have

$$\langle x|f\rangle = \int dx'\, \langle x|x'\rangle\langle x'|f\rangle \ , \tag{3.11}$$

which since this is to be true for all periodic functions, means that

$$\langle x|x'\rangle = \delta_L(x-x') \ . \tag{3.12}$$

The continuous set of position basis vectors are orthogonal and normalized in the sense (3.12), which is called δ-normalization. Moreover, one sees from Eq. (3.10) that the position vectors satisfy the *completeness relation*

$$I = \int dx\, |x\rangle\langle x| \ , \tag{3.13}$$

where I is the unit operator.

Working in the Fourier basis $|e_n\rangle$ is called using the *Fourier representation* of the vectors (functions), and working in the position basis $|x\rangle$ is called using the *position representation.* Projecting a vector $|f\rangle$ onto the Fourier basis gives the Fourier coefficients $c_n = \langle e_n|f\rangle$, and projecting onto the position basis gives the function values $f(x) = \langle x|f\rangle$. Just as for any two bases in a finite-dimensional vector space, the relation between the two representations is specified by the inner products

$$\langle x|e_n\rangle = e_n(x) = \frac{1}{\sqrt{L}}e^{ik_n x} \ . \tag{3.14}$$

Both bases satisfy a completeness relation,

$$I = \sum_{n=-\infty}^{\infty} |e_n\rangle\langle e_n| = \int dx\, |x\rangle\langle x| \ ; \tag{3.15}$$

applying these representations of the unit operator to an arbitrary vector $|f\rangle$ generates the expansions (3.9) and (3.10). The position-basis representation of the completeness relation is

$$\delta_L(x-x') = \langle x|I|x'\rangle = \sum_{n=-\infty}^{\infty} e_n(x)e_n^*(x) = \frac{1}{L}\sum_{n=-\infty}^{\infty} e^{i2\pi n(x-x')/L} \ , \tag{3.16}$$

and the Fourier-basis representation is

$$\delta_{nm} = \langle e_n|I|e_m\rangle = \int dx\, e_n^*(x) e_m(x) \ . \tag{3.17}$$

These are just different ways of writing the orthonormality conditions for the two bases, and they express the fact that the "matrix" of inner products, $\langle x|e_n\rangle = e_n(x)$, which does the transformation from the Fourier basis to the position basis, is unitary.

The use of position basis vectors is not essential, but it makes the notation more compact and easier to use, and it highlights the structure of the complex vector space of periodic functions.

4. THE FOURIER TRANSFORM

To eliminate the periodic structure in a localized function, we need to include even more Fourier components; indeed, it should be obvious that we have to include Fourier functions that are not periodic with period L. We can do this by considering the function $h_{a,L}(x)$ to be defined on the central interval $-L/2 \le x \le L/2$ and taking the limit $L \to \infty$ while keeping a fixed. The limit pushes all the bumps except the central one out beyond infinity, leaving a function $h_a(x)$ with a single bump of width a centered at the origin:

$$h_a(x) = \lim_{L\to\infty} h_{a,L}(x) = \begin{cases} 1/a\ , & \text{for } |x| \le a/2, \\ 0\ , & \text{for } |x| > a/2. \end{cases} \tag{4.1}$$

We now have to take the required limit in the Fourier series (2.7) and in the Fourier coefficient (2.10). Dealing with the Fourier series first, we write Eq. (2.7) as

$$h_{a,L}(x) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi/L} c_n e^{ik_n x} = \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi} \sqrt{L} c_n e^{ik_n x} \ . \tag{4.2}$$

Here $\Delta k = 2\pi/L$ is the difference between successive values of k_n. To deal with the Fourier coefficient (2.10), we define a function $\tilde{h}_a(k)$ of wave number k by

$$\tilde{h}_a(k) \equiv \lim_{L\to\infty} \sqrt{L} c_n = \lim_{L\to\infty} \sqrt{L} c_{kL/2\pi} \ . \tag{4.3}$$

In taking this limit, we write the Fourier coefficient as $c_n = c_{kL/2\pi}$ and think of it as a function of k instead of n. In taking the limit, both n and L go to infinity, with their ratio $n/L = k/2\pi$ held constant. What is happening is that as L gets big, more and more wavelengths for constant wave number $k = 2\pi n/L$ fit into an interval of length L, so n gets bigger, too. Taking the limit in Eq. (2.10) gives

$$\tilde{h}_a(k) = \frac{\sin(ka/2)}{ka/2} \ , \tag{4.4}$$

and the limit of Eq. (4.2) is

$$h_a(x) = \lim_{L\to\infty} \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi} \sqrt{L} c_n e^{ik_n x} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{h}_a(k) e^{ikx} \ . \tag{4.5}$$

The function $\tilde{h}_a(k)$ is called the *Fourier transform* of $h_a(x)$. Equation (4.5) gives the single-bump function $h_a(x)$ as a continuous superposition of Fourier components, the Fourier coefficient for each component now being given by the continuous function $\tilde{h}_a(k)$.

The lessons here are nearly the same as above. To represent a *single* bump of width a requires a continuous superposition of monochromatic waves. The Fourier transform $\tilde{h}_a(k)$ tells how much and with what phase each monochromatic wave contributes to the superposition [for the bump $h_a(x)$, the phase is zero for all Fourier components]. The Fourier transform is roughly constant for small wave numbers satisfying $|k|a \lesssim 2\pi$ and then falls off as $1/|k|$ for larger wave numbers. The Fourier components interfere constructively within the bump near

$x = 0$ and interfere destructively otherwise. The relation between the width of the function and the effective width of the Fourier transform is an expression of the uncertainty principle.

A corollary of this lesson occurs in the limit $a \to 0$. The "function" in this limit is called the δ-function:

$$\delta(x) \equiv \lim_{a\to 0} h_a(x) . \tag{4.6}$$

The "bump" in $\delta(x)$ has zero width, but such enormous height that the integral under it is 1. To represent $\delta(x)$ in terms of a Fourier transform, we need all Fourier components contributing to the superposition with Fourier coefficients $\lim_{a\to 0} \tilde{h}_a(k) = 1$:

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} . \tag{4.7}$$

The key property of the δ-function is that for any function $f(x)$, we have

$$\int_{-\infty}^{\infty} dx\, \delta(x) f(x) = \lim_{a\to 0} \int_{-\infty}^{\infty} dx\, h_a(x) f(x) = \lim_{a\to 0} \frac{1}{a} \int_{-a/2}^{a/2} dx\, f(x) = f(0) . \tag{4.8}$$

The limit in Eq. (4.6)—and, hence, the function $\delta(x)$ itself—only makes sense when one does an integral and takes the limit afterward.

It takes only a little work now to find the general relation between a function and its Fourier transform. We start with a periodic function $f_L(x)$, whose Fourier series is given by Eq. (2.7) and whose Fourier coefficients coefficients are given by Eq. (2.10). We then take the limit $L \to \infty$ so that the central interval $-L/2 \le x \le L/2$ occupies the entire real line. Following through the same steps as for the bump function $h_L(x)$, one sees that the resulting function $f(x)$ is related to the Fourier transform as in Eq. (4.5),

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{f}(k) e^{ikx} , \tag{4.9}$$

and the Fourier transform $\tilde{f}(k) \equiv \lim_{L\to\infty} \sqrt{L} c_n = \lim_{L\to\infty} \sqrt{L} c_{kL/2\pi}$ is now given by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, f(x) e^{-ikx} , \tag{4.10}$$

where we take the limit in Eq. (2.10). Equations (4.9) and (4.10) are called a *Fourier-transform pair. The Fourier transform is the most important integral transform in physics.*

A caution: Now that you have seen this "derivation" of the Fourier transform from the Fourier series, which though instructive, isn't really rigorous, you should put the derivation away in the corner of your mind and forget about it. *The derivation is not a tool for doing anything. The right mathematical tools are the Fourier series pair (2.7) and (2.10) or the Fourier transform pair (4.9) and (4.10); you should apply these two tools, not the derivation, when you need them.*

You will often see the Fourier transform in slightly different guises. The chief one of these is that the $1/2\pi$ in the k integral can be put in the x integral instead, or it can be distributed symmetrically as a $1/\sqrt{2\pi}$ in both the k and x integrals. I like the convention used here, because *all you have to remember is that the integration measure for wave number is always $dk/2\pi$* (or $d\omega/2\pi$ for a temporal Fourier transform); what this is telling us is that despite our use of wave number k or angular frequency ω, the natural variable in Fourier space is reduced wave number, $k/2\pi = 1/\lambda$, or frequency, $\nu = \omega/2\pi = 1/\tau$.

As another example of different guises, when dealing with time and angular frequency instead of position and wave number, physicists almost universally use the opposite sign convention in the complex exponentials, i.e.,

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \tilde{f}(\omega) e^{-i\omega t} \qquad \text{and} \qquad \tilde{f}(\omega) = \int_{-\infty}^{\infty} dt\, f(t) e^{i\omega t} \tag{4.11}$$

(see footnote 1); engineers, on the other hand, tend to stick with the original sign convention when dealing with time and frequency. Another notation you will often see leaves the tilde off the Fourier transform, letting

Properties of the Fourier transform

1. *Linearity*:	$h(x) = \alpha f(x) + \beta g(x)$	$\Longleftrightarrow$	$\tilde{h}(k) = \alpha\tilde{f}(k) + \beta\tilde{g}(k)$
2. *Conjugation*:	$h(x) = f^*(x)$	$\Longleftrightarrow$	$\tilde{h}(-k) = \tilde{f}^*(k)$
3. *Reality*:	$f(x) = f^*(x)$	$\Longleftrightarrow$	$\tilde{f}(-k) = \tilde{f}^*(k)$
4. *Parity*:	$h(x) = f(-x)$	$\Longleftrightarrow$	$\tilde{h}(k) = \tilde{f}(-k)$
5. *Even-odd symmetry*:	$f(x) = \pm f(-x)$	$\Longleftrightarrow$	$\tilde{f}(k) = \pm\tilde{f}(-k)$
6. *Product*:	$h(x) = f(x)g(x)$	$\Longleftrightarrow$	$\tilde{h}(k) = \int_{-\infty}^{\infty}\frac{dk'}{2\pi}\,\tilde{f}(k')\tilde{g}(k-k')$
7. *Convolution*:	$h(x) = \int_{-\infty}^{\infty} dx'\, f(x')g(x-x')$	$\Longleftrightarrow$	$\tilde{h}(k) = \tilde{f}(k)\tilde{g}(k)$
8. *Spatial derivative*:	$h(x) = \frac{d^n f(x)}{dx^n}$	$\Longleftrightarrow$	$\tilde{h}(k) = (ik)^n\tilde{f}(k)$
9. *Fourier derivative*:	$h(x) = (-ix)^n f(x)$	$\Longleftrightarrow$	$\tilde{h}(k) = \frac{d^n\tilde{f}(k)}{dk^n}$
10. *Completeness*:	$f(x) = \delta(x-x') = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,e^{ik(x-x')}$	$\Longleftrightarrow$	$\tilde{f}(k) = e^{-ikx'}$

the dependent variable, x or k, indicate whether one is dealing with the function $f(x)$ or its Fourier transform $\tilde{f}(k)$. Yet another possibility, used by Boas to distinguish further a function from its Fourier transform, is to use different letters for a function (f in Boas) and its Fourier transform (g in Boas); this notation, initially attractive for its clarity, quickly becomes untenable since one doesn't want to call all functions f. Boas also uses a different letter, α, for the Fourier variable k, but this is decidedly heterodox notation.

The table lists properties of the Fourier transform. Notice that the reality property is a special case of conjugation and that even-odd symmetry is a special case of parity. Combining reality and even-odd symmetry, one has that a real, even function has a Fourier transform that is real and even and a real, odd function has a Fourier transform that is pure imaginary and odd. The spatial-derivative property says that the derivative operator, d/dx, in the spatial domain corresponds to multiplication by ik in the Fourier domain; this is perhaps the most useful property of the Fourier transform, since it turns linear differential equations in space into algebraic equations in the Fourier domain.

Completeness of the Fourier functions is the statement that any function can be expanded in terms of the Fourier functions e^{ikx}, and this means that for any function $f(x)$,

$$f(x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\tilde{f}(k)e^{ikx} = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,e^{ikx}\int_{-\infty}^{\infty}dx'\,f(x')e^{-ikx'} = \int_{-\infty}^{\infty}dx'\,f(x')\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,e^{ik(x-x')}\ . \tag{4.12}$$

The only way this can be true for all functions is to have the completeness property,

$$\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,e^{ik(x-x')} = \delta(x-x')\ , \tag{4.13}$$

which is the same as the definition of the δ-function in Eq. (4.7). The inverse of the completeness property is the physical statement that the Fourier transform of a plane wave $e^{ik'x}$ is a δ-function, i.e., has only one Fourier component:

$$f(x) = e^{ik'x} \quad\Longleftrightarrow\quad \tilde{f}(k) = \int_{-\infty}^{\infty}dx\,e^{i(k'-k)x} = 2\pi\delta(k'-k)\ . \tag{4.14}$$

The only properties that might present some difficulty are the product and convolution properties, but even these are easy to derive using the completeness property (check it!). The product and convolution properties are inverses of one another. The product property says that the Fourier transform of a product of two functions is the *convolution* of the Fourier transforms of those two functions, whereas the convolution property says that the Fourier transform of the convolution of two functions is the product of the Fourier transforms of the two functions. These two properties are important in the theory of dispersion and in the theory of signal processing.

An important special case of the product property occurs when $g(x) = f^*(x)$, so that $h(x) = |f(x)|^2$:

$$\int_{-\infty}^{\infty} dx\, |f(x)|^2 e^{ikx} = \tilde{h}(k) = \int_{-\infty}^{\infty} \frac{dk'}{2\pi}\, \tilde{f}(k')\tilde{f}^*(k'-k)\ . \tag{4.15}$$

This special case is called *Parseval's relation.* When evaluated at $k = 0$, Parseval's relation reduces to

$$\int_{-\infty}^{\infty} dx\, |f(x)|^2 = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, |\tilde{f}(k)|^2\ . \tag{4.16}$$

We now give some examples of Fourier-transform pairs, together with comments about their significance.

$$\begin{aligned}
f(x) &= \begin{cases} e^{i\kappa x}\ , & \text{for } |x| \le a/2 \\ 0\ , & \text{for } |x| > a/2 \end{cases} &&\Longleftrightarrow\quad \tilde{f}(k) = \frac{\sin[(k-\kappa)a/2]}{(k-\kappa)/2} \\
f(x) &= \begin{cases} \cos\kappa x\ , & \text{for } |x| \le a/2 \\ 0\ , & \text{for } |x| > a/2 \end{cases} &&\Longleftrightarrow\quad \tilde{f}(k) = \frac{1}{2}\left(\frac{\sin[(k-\kappa)a/2]}{(k-\kappa)/2} + \frac{\sin[(k+\kappa)a/2]}{(k+\kappa)/2}\right) \\
f(x) &= \begin{cases} \sin\kappa x\ , & \text{for } |x| \le a/2 \\ 0\ , & \text{for } |x| > a/2 \end{cases} &&\Longleftrightarrow\quad \tilde{f}(k) = \frac{1}{2i}\left(\frac{\sin[(k-\kappa)a/2]}{(k-\kappa)/2} - \frac{\sin[(k+\kappa)a/2]}{(k+\kappa)/2}\right)
\end{aligned} \tag{4.17}$$

Notice that the second and third of these pairs follows from the first by using Euler's relation on $e^{i\kappa x}$. The first is an example of what we set out to do in this document, putting rapid oscillations within an envelope of limited spatial extent: when $\kappa a \gg 2\pi$, the function $f(x)$ is a wave packet of unit height and spatial extent a with wave number κ, unit height. The Fourier transform is concentrated near $k = \kappa$: for $|k-\kappa|a \ll 1$, its value is approximately a, and for $|k-\kappa|a \gg 1$, it falls off as $2/|k-\kappa|$.

In the following examples, $H(x)$ stands for the *Heaviside step function*

$$H(x) = \begin{cases} 0\ , & \text{for } x < 0, \\ 1\ , & \text{for } x > 0. \end{cases} \tag{4.18}$$

$$\begin{aligned}
f(x) &= H(x)e^{-\gamma x}e^{i\kappa x} = \begin{cases} 0\ , & \text{for } x < 0 \\ e^{-\gamma x}e^{i\kappa x}\ , & \text{for } x > 0 \end{cases} &&\Longleftrightarrow\quad \tilde{f}(k) = \frac{1}{i}\frac{1}{k-\kappa-i\gamma} = \frac{1}{i}\frac{k-\kappa+i\gamma}{(k-\kappa)^2+\gamma^2} \\
f(x) &= H(-x)e^{\gamma x}e^{i\kappa x} = \begin{cases} e^{\gamma x}e^{i\kappa x}\ , & \text{for } x < 0 \\ 0\ , & \text{for } x > 0 \end{cases} &&\Longleftrightarrow\quad \tilde{f}(k) = i\frac{1}{k-\kappa+i\gamma} = i\frac{k-\kappa-i\gamma}{(k-\kappa)^2+\gamma^2} \\
f(x) &= e^{-\gamma|x|}e^{i\kappa x} &&\Longleftrightarrow\quad \tilde{f}(k) = \frac{2\gamma}{(k-\kappa)^2+\gamma^2}\ , \\
f(x) &= e^{-\gamma|x|}\cos\kappa x &&\Longleftrightarrow\quad \tilde{f}(k) = \frac{\gamma}{(k-\kappa)^2+\gamma^2} + \frac{\gamma}{(k+\kappa)^2+\gamma^2} \\
f(x) &= e^{-\gamma|x|}\sin\kappa x &&\Longleftrightarrow\quad \tilde{f}(k) = \frac{1}{i}\left(\frac{\gamma}{(k-\kappa)^2+\gamma^2} - \frac{\gamma}{(k+\kappa)^2+\gamma^2}\right)
\end{aligned} \tag{4.19}$$

All these transform pairs follow from the first one in the list, which written as an inverse Fourier transform, becomes

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} dk\, \frac{e^{ikx}}{k-\kappa-i\gamma} = H(x)e^{-\gamma x}e^{i\kappa x}\ . \tag{4.20}$$

This is a special case of some very general results that you will learn when you study contour integrals in complex analysis.

Our last set of transform pairs involves Fourier transforms of Gaussian functions.

$$\begin{aligned}
f(x) &= \sqrt{\frac{\gamma}{\pi}} e^{-\gamma x^2} e^{i\kappa x} &\iff& \quad \tilde{f}(k) = e^{-(k-\kappa)^2/4\gamma} \\
f(x) &= \sqrt{\frac{\gamma}{\pi}} e^{-\gamma x^2} \cos\kappa x &\iff& \quad \tilde{f}(k) = \frac{1}{2}\left(e^{-(k-\kappa)^2/4\gamma} + e^{-(k+\kappa)^2/4\gamma}\right) \\
f(x) &= \sqrt{\frac{\gamma}{\pi}} e^{-\gamma x^2} \cos\kappa x &\iff& \quad \tilde{f}(k) = \frac{1}{2i}\left(e^{-(k-\kappa)^2/4\gamma} - e^{-(k+\kappa)^2/4\gamma}\right)
\end{aligned} \tag{4.21}$$

You should notice that the spatial Gaussian here is normalized to unity, i.e.,

$$\int_{-\infty}^{\infty} dx \sqrt{\frac{\gamma}{\pi}} e^{-\gamma x^2} = \tilde{f}(\kappa) = 1 \; . \tag{4.22}$$

In all three sets of transform pairs, the function $f(x)$ can be thought of as a wave packet. Only in the first set is the wave packet strictly localized, but in the second two sets, the wave packet goes to zero either exponentially or as a Gaussian, which is fast enough that the packet can be regarded as localized. The relation between the width of the wave packet and the width of its Fourier transform is an example of the uncertainty principle.

5. THE δ-FUNCTION

The δ-function plays a crucial role in Fourier-transform theory. Its most useful representation, given in Eq. (4.7) and repeated here,

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, e^{ikx} \; , \tag{5.1}$$

says that it is the function whose Fourier transform is a constant. The integral in Eq. (5.1) doesn't converge in any conventional sense, but what it conveys is that for $x \neq 0$, the oscillations of e^{ikx} average the integral over k to zero, and for $x = 0$, the integral is so divergent as to provide the peak of the δ-function at $x = 0$. The δ-function can also be thought of as the limit of a function with a narrow width and a high central peak that is normalized so that the integral over the function is 1. Both these ways of thinking about the δ-function indicate that it is not a real function, but rather is a mathematical tool that can be used inside integrals that tame the divergences. Notice that the δ-function, no matter how you think about it, is real and even, and this is reflected in that its (constant) Fourier transform is real and even.

The Fourier transform pairs provide examples of highly peaked functions that limit to the δ-function and how their Fourier transforms limit to 1:

$$\begin{array}{llll}
h_a(x) = \begin{cases} 1/a \, , & \text{for } |x| \le a/2 \\ 0 \, , & \text{for } |x| > a/2 \end{cases} & \delta(x) = \lim\limits_{a\to 0} h_a(x) & \tilde{h}_a(k) = \dfrac{\sin(ka/2)}{ka/2} & \lim\limits_{a\to 0} \tilde{h}_a(k) = 1 \\
h_\gamma(x) = \dfrac{\gamma}{2} e^{-\gamma|x|} & \delta(x) = \lim\limits_{\gamma\to\infty} \dfrac{\gamma}{2} e^{-\gamma|x|} & \tilde{h}_\gamma(k) = \dfrac{\gamma^2}{k^2+\gamma^2} & \lim\limits_{\gamma\to\infty} \tilde{h}_\gamma(k) = 1 \\
h_\gamma(x) = \sqrt{\dfrac{\gamma}{\pi}} e^{-\gamma x^2} & \delta(x) = \lim\limits_{\gamma\to\infty} \sqrt{\dfrac{\gamma}{\pi}} e^{-\gamma x^2} & \tilde{h}_\gamma(k) = e^{-k^2/4\gamma} & \lim\limits_{\gamma\to\infty} \tilde{h}_\gamma(k) = 1
\end{array} \tag{5.2}$$

To drive the point home, let's write out the Fourier transform for the latter two of these:

$$\begin{aligned}
\frac{\gamma}{2} e^{-\gamma|x|} &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, \frac{\gamma^2}{k^2+\gamma^2} \, e^{ikx} \; , \\
\sqrt{\frac{\gamma}{\pi}} e^{-\gamma x^2} &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, e^{-k^2/4\gamma} e^{ikx} \; .
\end{aligned} \tag{5.3}$$

The $\gamma \to \infty$ limit of both of these gets you to the divergent Fourier transform (5.1) for $\delta(x)$.

The key property of the δ-function, expressed in the completeness property (4.12), is that when integrated against any function, the δ-function picks out the value of the function at the peak of the δ-function:

$$f(x) = \int_{-\infty}^{\infty} dx'\, f(x')\delta(x' - x)\ . \tag{5.4}$$

Equation (4.8) shows how this property comes from taking a limit on a highly peaked function and thus illustrates what the limits in Eq. (5.2) actually mean: the limit only makes sense when integrating over the δ-function, and then the integral should be done first and the limit taken afterward. That Eq. (5.4) holds for any function $f(x)$, means that we can also write

$$f(x) = \int_{x-\epsilon}^{x+\epsilon} dx'\, f(x')\delta(x' - x)\ , \tag{5.5}$$

which holds no matter how small ϵ is. This result can also be regarded as the defining property of the δ-function.

There is another way to represent the δ-function, as the derivative of the Heaviside function:

$$\delta(x) = \frac{dH(x)}{dx} = H'(x)\ . \tag{5.6}$$

Intuitively, this works because the derivative of the Heaviside function is zero everywhere except at $x = 0$, where it is infinite. More generally, the derivative of a function at any discontinuity in the function, though ill defined by any standard definition, can be treated as a δ-function within integrals. Thus we can justify Eq. (5.6) by integrating over $H'(x)$ and doing an integration by parts:

$$\begin{aligned}\int_{-\epsilon}^{\epsilon} dx\, f(x)H'(x) &= f(x)H(x)\big|_{x=-\epsilon}^{x=\epsilon} - \int_{-\epsilon}^{\epsilon} dx\, f'(x)H(x)\\ &= f(\epsilon) - \int_{0}^{\epsilon} dx\, f'(x)\\ &= f(\epsilon) - [f(\epsilon) - f(0)]\\ &= f(0)\ .\end{aligned} \tag{7}$$

We can also justify Eq. (5.6) by using the first Fourier transform pair in Eq. (4.19). Let $h_\gamma(x) = H(x)e^{-\gamma x}$, so that $H(x) = \lim_{\gamma\to 0} h_\gamma(x)$. The Fourier transform of $h_\gamma(x)$ is $\tilde{h}_\gamma(k) = 1/i(k - i\gamma)$, so the Fourier transform of $h'_\gamma(x)$ is $ik\tilde{h}_\gamma(x) = k/(k - i\gamma)$, which limits to 1 as γ goes to zero.

There is occasional need for derivatives of the δ-function, so let's take a look at the first derivative $\delta'(x)$. Thinking in terms of limits of highly peaked functions, we can see that $\delta'(x)$ is the limit of a function with two high peaks, a positive peak just to the left of $x = 0$ and a negative peak just to right of $x = 0$. If we think in terms of Fourier transforms, the Fourier transform of $\delta'(x)$ is equal to ik. The best way to see how $\delta'(x)$ can be used is to see what it does within an integral:

$$\int_{-\epsilon}^{\epsilon} dx\, f(x)\delta'(x) = f(x)\delta(x)\big|_{x=-\epsilon}^{x=\epsilon} - \int_{-\epsilon}^{\epsilon} dx\, f'(x)\delta(x) = -f'(0)\ . \tag{8}$$

A sometimes useful formula, which is a bit surprising, is that

$$x\delta'(x) = -\delta(x)\ . \tag{5.9}$$

The easiest way to see this is, again, to look at an integral:

$$\int_{-\epsilon}^{\epsilon} dx\, f(x)x\delta'(x) = -\left.\frac{d}{dx}xf(x)\right|_{x=0} = -f(0)\ . \tag{5.10}$$

One can also think in terms of Fourier transforms. Multiplying by $-ix$ in the spatial domain is the same as differentiating with respect to k in the Fourier domain; since the Fourier transform of $\delta'(x)$ is ik, differentiating with respect to k gets us a constant Fourier transform, i, which is the Fourier transform of $-ix\delta'(x)$ and also the transform of $i\delta(x)$. So $\delta(x) = -x\delta'(x)$.

Properties of the δ-function

$$f(x) = \int_{x-\epsilon}^{x+\epsilon} dx'\, f(x')\delta(x'-x) \quad \text{for any value of } \epsilon \text{, no matter how small ,}$$

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} \qquad \text{This means that the Fourier transform } \tilde{\delta}(k) = 1.$$

$$\delta(x) = \lim_{\epsilon\to 0} h_\epsilon(x) \;\text{ where }\; h_\epsilon(x) = \begin{cases} 1/\epsilon\,, & \text{for } |x| \le \epsilon/2 \\ 0\,, & \text{for } |x| > \epsilon/2 \end{cases}$$

$$\delta(x) = \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\, e^{-|x|/\epsilon}$$

$$\delta(x) = \lim_{\epsilon\to 0} \frac{1}{\sqrt{\pi\epsilon}}\, e^{-x^2/\epsilon}$$

$$\delta(x) = H'(x)$$

$$\delta(x) = -x\delta'(x)$$

$$\delta(bx) = \delta(|b|x) = \frac{1}{|b|}\delta(x)$$

$$\delta\big(f(x)\big) = \sum_j \delta\big(f'(x_j)(x-x_j)\big) = \sum_j \frac{1}{|f'(x_j)|}\delta(x-x_j) \quad \text{where the sum runs over } x_j \text{ such that } f(x_j)=0$$

Since the main use of δ-functions is in integrals, we need to know how to change it when its argument changes. Let's first consider $\delta(bx)$, where $b > 0$; now consider the integral

$$\int_{-\epsilon}^{\epsilon} dx\, f(x)\delta(bx) = \int_{-b\epsilon}^{b\epsilon} \frac{dy}{b}\, f(y/b)\delta(y) = \frac{f(0)}{b}\ , \tag{5.11}$$

where we make the change of integration variable $y = bx$. This shows us that $\delta(bx) = \delta(x)/b$. The way we did the change of variables requires that $b > 0$, but the evenness of the δ-function means that

$$\delta(bx) = \delta(|b|x) = \frac{1}{|b|}\delta(x)\ , \tag{5.12}$$

which applies whether b is positive or negative.

A more general result that follows from Eq. (5.12) concerns how to handle $\delta\big(f(x)\big)$. We get at this by noting that $\delta\big(f(x)\big)$ is nonzero only at the zeros x_j of $f(x)$, near which it can be expanded as $f(x) = f'(x_j)(x - x_j)$. So we have

$$\delta\big(f(x)\big) = \sum_j \delta\big(f'(x_j)(x-x_j)\big) = \sum_j \frac{1}{|f'(x_j)|}\delta(x-x_j)\ , \tag{5.13}$$

where the sum is over the zeros of $f(x)$. Messing up changes of variables in the argument of a δ-function is one of the most common mistakes in using the δ-function.

One thing we have left hanging is the relation between the periodic δ-function $\delta_L(x)$ and the $\delta(x)$, so let's figure that out. We begin with

$$\delta_L(x) = \int dx'\, \delta(x'-x)\delta_L(x') = \sum_{n=-\infty}^{\infty} \int_{-L/2-nL}^{L/2-nL} dx'\, \delta(x'-x)\delta_L(x')\ , \tag{5.14}$$

where in the second form, we have divided up the integral on the entire real line into integrals over all the segments of length L. Now let's change the integral variable to $y = x - nL$, so that all the integrals are over the same interval,

$$\delta_L(x) = \sum_{n=-\infty}^{\infty} \int_{-L/2}^{L/2} dy\, \delta(y+nL-x)\delta_L(y+nL) = \sum_{n=-\infty}^{\infty} \int_{-L/2}^{L/2} dy\, \delta(y+nL-x)\delta_L(y) \tag{5.15}$$

where the second equality again follows from the periodicity of the periodic δ-function. Now we use the integration property of $\delta_L(y)$ to get the result:

$$\frac{1}{L}\sum_{n=-\infty}^{\infty} e^{i2\pi nx/L} = \delta_L(x) = \sum_{n=-\infty}^{\infty} \delta(x-nL)\ . \tag{5.16}$$

This is pretty much what one would expect: $\delta_L(x)$ consists of a series of δ-peaks at the points $x = nL$. The Fourier transform of the periodic δ-function has two nice forms:

$$\begin{aligned}
\tilde{\delta}_L(k) &= \int_{-\infty}^{\infty} dx\, \delta_L(x) e^{-ikx} = \frac{1}{L}\sum_{n=-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} dx\, e^{i(k_n-k)x}}_{=\,2\pi\delta(k-k_n)} = \frac{2\pi}{L}\sum_{n=-\infty}^{\infty} \delta\left(k - \frac{2\pi n}{L}\right) \\
\tilde{\delta}_L(k) &= \int_{-\infty}^{\infty} dx\, \delta_L(x) e^{-ikx} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dx\, \delta(x-nL) e^{-ikx} = \sum_{n=-\infty}^{\infty} e^{-iknL}
\end{aligned} \tag{5.17}$$

You should notice the symmetry under exchange of x and k in the forms for $\delta_L(x)$ and $\tilde{\delta}_L(k)$.

The δ-function is a huge convenience. It makes our lives easier by allowing us to write equations for things where without it, we would have always to be talking about taking limits on highly peaked functions. Use of the δ-function means that we already know how to take those limits within integrals, so we don't have to be referring to them constantly. But using the δ-function requires that you think, because sometimes using it can blow up in your face. It's a little like electricity: a huge convenience, which we couldn't do without, but that doesn't mean you go sticking a screwdriver into an electrical outlet. Problems with the δ-function arise when its peak coincides with a place where other functions have discontinuities or even δ-like divergences. When that happens, you can get nonsense for an answer and have to retreat to modeling the discontinuities and divergences by actual functions; that should tell you how properly to use the δ-function in the problem at hand.

6. FOURIER TRANSFORMS AS A COMPLEX VECTOR SPACE (Advanced Topic)

The functions on the real line make up a complex vector with the inner product

$$(g,f) = \langle g|f\rangle = \int_{-\infty}^{\infty} dx\, g^*(x) f(x)\ . \tag{6.1}$$

Just as in Sec. 3, we have to restrict the vector space to functions whose squared magnitude is finite,

$$\langle f|f\rangle = \int_{-\infty}^{\infty} dx\, |f(x)|^2 < \infty\ ; \tag{6.2}$$

the resulting vector space is called the space of square-integrable functions on the real line.

We can plug Fourier transforms into the definition of the inner product to convert it from the spatial domain to the Fourier domain:

$$\begin{aligned}
\langle g|f\rangle &= \int_{-\infty}^{\infty} dx\, g^*(x) f(x) \\
&= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dk'}{2\pi}\, \tilde{g}^*(k') e^{-ik'x} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{f}(k) e^{ikx} \\
&= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{dk'}{2\pi}\, \tilde{g}^*(k')\tilde{f}(k) \underbrace{\int_{-\infty}^{\infty} dx\, e^{i(k-k')x}}_{=\,2\pi\delta(k-k')} \\
&= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{g}^*(k)\tilde{f}(k)\ .
\end{aligned} \tag{6.3}$$

The Fourier-domain version of the inner product looks just the same as the spatial version except for that factor of $1/2\pi$, which we're going to take account of as we go along. Notice that when $g = f$, this result simplifies to Eq. (4.16).

Just as we did for periodic functions, we are tempted to regard function values, $f(x)$ in space and $\tilde{f}(k)$ in the Fourier domain, as continuous "components" of the vector $|f\rangle$, i.e., $f(x) = \langle x|f\rangle$ and $\tilde{f}(k) = \langle k|f\rangle$. To get this to work, let's make the position basis vectors $|x\rangle$ be δ-normalized and complete[4]:

$$\langle x|x'\rangle = \delta(x - x') , \qquad I = \int_{-\infty}^{\infty} dx\,|x\rangle\langle x| . \tag{6.4}$$

These are consistent because

$$\langle x|x'\rangle = \langle x|I|x'\rangle = \int_{-\infty}^{\infty} dx''\,\langle x|x''\rangle\langle x''|x'\rangle = \int_{-\infty}^{\infty} dx''\,\delta(x - x'')\delta(x'' - x') = \delta(x - x') . \tag{6.5}$$

A vector $|f\rangle$ has the position representation

$$|f\rangle = I|f\rangle = \int_{-\infty}^{\infty} dx\,|x\rangle\langle x|f\rangle = \int_{-\infty}^{\infty} dx\,f(x)|x\rangle . \tag{6.6}$$

The transformation from the position basis vectors to the Fourier basis vectors is given by the Fourier transform:

$$\langle x|f\rangle = f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\tilde{f}(k)e^{ikx} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\,e^{ikx}\langle k|f\rangle . \tag{6.7}$$

If we let $\langle x|k\rangle = e^{ikx}$ describe the transformation between position and Fourier basis vectors, then the Fourier transform (6.7) becomes

$$\langle x|f\rangle = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\langle x|k\rangle\langle k|f\rangle , \tag{6.8}$$

and we can nail things down by realizing that this is the completeness property for the Fourier-domain vectors,

$$I = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\,|k\rangle\langle k| . \tag{6.9}$$

The appropriate δ-normalization now follows from

$$\langle k|k'\rangle = \langle k|I|k'\rangle = \int_{-\infty}^{\infty} dx\,\langle k|x\rangle\langle x|k'\rangle = \int_{-\infty}^{\infty} dx\,e^{i(k'-k)x} = 2\pi\delta(k - k') = \delta\!\left(\frac{k}{2\pi} - \frac{k'}{2\pi}\right) . \tag{6.10}$$

Here is a summary. The square-integrable functions on the real line are vectors in a complex vector space. The values of a function in space are the components of the vector in the position basis, $f(x) = \langle x|f\rangle$, and the values of the Fourier transform are the components of the vector in the Fourier basis, $\tilde{f}(k) = \langle k|f\rangle$. The two bases are related by the inner products

$$\langle x|k\rangle = e^{ikx} , \tag{6.11}$$

and they have the following properties:

$$\begin{array}{lll} \delta\text{-normalization:} & \langle x|x'\rangle = \delta(x - x') & \langle k|k'\rangle = 2\pi\delta(k - k') = \delta\!\left(\dfrac{k}{2\pi} - \dfrac{k'}{2\pi}\right) \\[2ex] \text{Completeness:} & I = \displaystyle\int_{-\infty}^{\infty} dx\,|x\rangle\langle x| & I = \displaystyle\int_{-\infty}^{\infty} \frac{dk}{2\pi}\,|k\rangle\langle k| \end{array} \tag{6.12}$$

[4] The vector space of functions here, i.e., the space of square-integrable functions on the real line, is not the same as the vector space of square-integrable periodic functions considered in Sec. 3, so the kets $|x\rangle$ here are *not* the same as the ones in Sec. 3.

The Fourier transform is the basis change from position to Fourier basis:

$$
\begin{aligned}
\langle x|f\rangle = \langle x|I|f\rangle &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\langle x|k\rangle\langle k|f\rangle \int_{-\infty}^{\infty} \frac{dk}{2\pi}\,e^{ikx}\langle k|f\rangle\;, \\
\langle k|f\rangle = \langle k|I|f\rangle &= \int_{-\infty}^{\infty} dx\,\langle k|x\rangle\langle x|f\rangle \int_{-\infty}^{\infty} dx\,e^{-ikx}\langle x|f\rangle\;.
\end{aligned}
\tag{6.13}
$$

The two forms of the inner product follow from using the completeness properties:

$$
\begin{aligned}
\langle g|f\rangle = \langle g|I|f\rangle &= \int_{-\infty}^{\infty} dx\,\langle g|x\rangle\langle x|f\rangle\;, \\
\langle g|f\rangle = \langle g|I|f\rangle &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\langle g|k\rangle\langle k|f\rangle\;.
\end{aligned}
\tag{6.14}
$$

Notice that as promised earlier, the difference between the position and Fourier domains is that we work with x in the position domain and $k/2\pi$ in the Fourier domain.

The reason this vector space of square-integrable functionsn is important is that in real problems, we are often confronted with a linear differential or partial-differential equation. The linear differential operator in this equation becomes a Hermitian (self-adjoint) linear operator in the space of square-integrable functions, and we solve for the complete set of orthonormal eigenfunctions (eigenvectors) and eigenvalues. Let's suppose, then, that we have an orthonormal set of vectors $|f_n\rangle$, for $n = 0, \ldots, \infty$, perhaps arising from some such aproblem; the corresponding orthonormal functions are $f_n(x) = \langle x|f_n\rangle$, and their Fourier transforms are $\tilde{f}_n(k) = \langle k|f_n\rangle$. The orthonormality property of these functions is

$$
\delta_{nm} = \langle f_n|f_m\rangle = \int_{-\infty}^{\infty} dx\,\langle f_n|x\rangle\langle x|f_m\rangle = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\langle f_n|k\rangle\langle k|f_m\rangle\;.
\tag{6.15}
$$

and their completeness is expressed as

$$
I = \sum_{n=0}^{\infty} |f_n\rangle\langle f_n|\;,
\tag{6.16}
$$

which has the position-basis and Fourier-basis forms

$$
\begin{aligned}
\delta(x-x') = \langle x|I|x'\rangle &= \sum_{n=0}^{\infty} \langle x|f_n\rangle\langle f_n|x'\rangle\;, \\
2\pi\delta(k-k') = \langle k|I|k'\rangle &= \sum_{n=0}^{\infty} \langle k|f_n\rangle\langle f_n|k'\rangle\;.
\end{aligned}
\tag{6.17}
$$

We can easily go back and forth between the position and Fourier bases and the new basis:

$$
\begin{aligned}
\langle x|f\rangle &= \sum_{n=0}^{\infty} \langle x|f_n\rangle\langle f_n|f\rangle\;, & \langle k|f\rangle &= \sum_{n=0}^{\infty} \langle k|f_n\rangle\langle f_n|f\rangle\;, \\
\langle f_n|f\rangle &= \int_{-\infty}^{\infty} dx\,\langle f_n|x\rangle\langle x|f\rangle\;, & \langle f_n|k\rangle &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\langle f_n|k\rangle\langle k|f\rangle\;.
\end{aligned}
\tag{6.18}
$$