Phys 366 **Mathematical Methods of Physics** **Fall 2015**

Quiz 4 **2015 November 3**
(100 points)

Problem 1 (100 points) The quadrupole-moment tensor

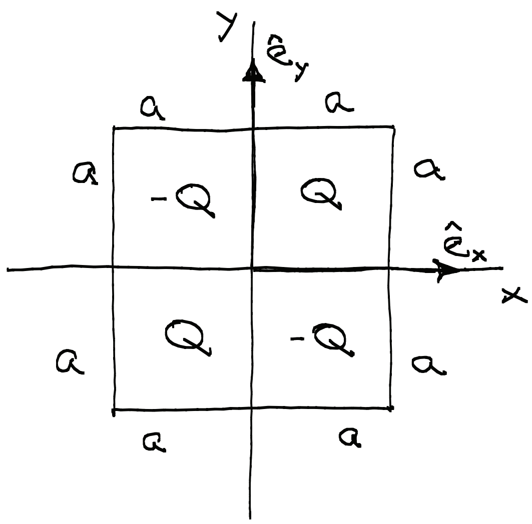
$$\mathbf{Q} = \sum_{j,k} Q_{jk} \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k ,$$

has components given by

$$Q_{jk} = \int d^3r \, \rho(\mathbf{r})(3x_j x_k - r^2 \delta_{jk}) ,$$

where $\rho(\mathbf{r})$ is the electric charge density.

This problem deals with a charge distribution that consists of four square tiles lying in the x-y plane. The tiles have sides of length a, and each has total charge $\pm Q$, which is spread uniformly over the surface of the tile. The positively and negatively charged tiles alternate in the four quadrants, as shown in the drawing. It is easy to see that this charge distribution has zero total charge and zero dipole-moment vector.

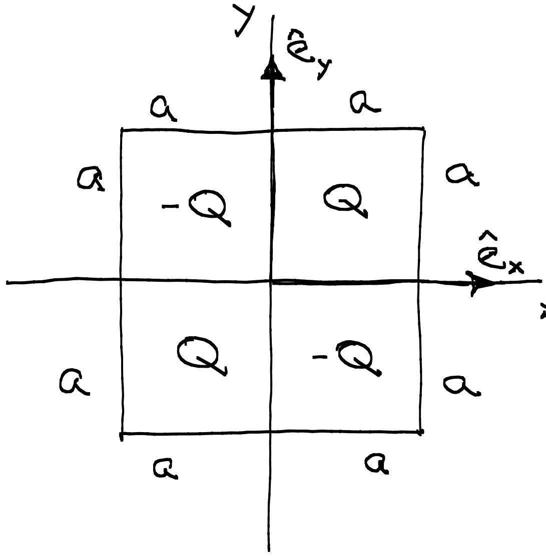


(a) (35 points) *Find* the components Q_{jk} of the quadrupole-moment tensor.

(b) (35 points) *Give* the eigenvalues (principal components) and eigenvectors (principal directions) of the quadrupole-moment tensor.

(c) (30 points) *Find* the orthogonal matrix that transforms from the original Cartesian basis vectors $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, $\hat{\mathbf{e}}_z$ to basis vectors $\hat{\mathbf{e}}'_x$, $\hat{\mathbf{e}}'_y$, $\hat{\mathbf{e}}'_z$ that are the eigenvectors (principal directions) of the quadrupole-moment tensor. *Demonstrate* that the quadruple-moment tensor transforms from the components of part (a) to the diagonal form of part (b).

Problem 1



$$Q_{jk} = \int dq\,(3x_j x_k - r^2\delta_{jk})$$

(a) $\int dq\, r^2 = 0$ ← the positive and negative tiles cancel each other.

$$\int dq\, x^2 = \int dq\, y^2 = 0$$

$$\int dq\, z^2 = \int dq\, xz = \int dq\, yz = 0$$ ← all the charge is at $z=0$

$$\int dq\, xy = \frac{Q}{a^2}\int_0^a dx \int_0^a dy\, xy + \frac{Q}{a^2}\int_{-a}^0 dx \int_{-a}^0 dy\, xy - \frac{Q}{a^2}\int_{-a}^0 dx \int_0^a dy\, xy - \frac{Q}{a^2}\int_0^a dx \int_{-a}^0 dy\, xy$$

all four contributions are equal

$$= \frac{4Q}{a^2}\underbrace{\int_0^a dx\, x}_{\frac{1}{2}a^2}\ \underbrace{\int_0^a dy\, y}_{\frac{1}{2}a^2}$$

$$\int dq\, xy = Qa^2$$

You should notice that $dq = \sigma\, da = \sigma\, dx\, dy$ in this problem, where σ is the surface charge density and we must $dx > 0$ and $dy > 0$, i.e., integration in the positive direction.

Positive tiles: $\sigma = \frac{Q}{a^2}$, $dq = \frac{Q}{a^2}dx\,dy$

Negative tiles: $\sigma = -\frac{Q}{a^2}$, $dq = -\frac{Q}{a^2}dx\,dy$

$$\Rightarrow \overleftrightarrow{Q} = 3Qa^2(\hat{e}_x \otimes \hat{e}_y + \hat{e}_y \otimes \hat{e}_x) \longleftrightarrow 3Qa^2\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) The eigenvectors are obvious either by symmetry or by previously diagonalizing this tensor:

① $\hat{e}'_x = \frac{1}{\sqrt{2}}(\hat{e}_x + \hat{e}_y) \longleftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$ Eigenvalue $3Qa^2$

② $\hat{e}'_y = \frac{1}{\sqrt{2}}(-\hat{e}_x + \hat{e}_y) \longleftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}$ Eigenvalue $-3Qa^2$

③ $\hat{e}'_z = \hat{e}_z \longleftrightarrow \begin{pmatrix}0\\0\\1\end{pmatrix}$ Eigenvalue 0

Verification:

① $$\overleftrightarrow{Q}\cdot\hat{e}'_x = \frac{1}{\sqrt{2}}\Big(\underbrace{\overleftrightarrow{Q}\cdot\hat{e}_x}_{3Qa^2\hat{e}_y} + \underbrace{\overleftrightarrow{Q}\cdot\hat{e}_y}_{3Qa^2\hat{e}_x}\Big) = 3Qa^2\,\hat{e}'_x$$

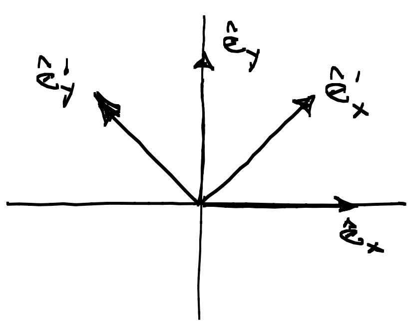
$$\longleftrightarrow 3Qa^2\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = 3Qa^2\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

① $$\overleftrightarrow{Q}\cdot\hat{e}'_y = \frac{1}{\sqrt{2}}\Big(-\underbrace{\overleftrightarrow{Q}\cdot\hat{e}_x}_{-3Qa^2\hat{e}_y} + \underbrace{\overleftrightarrow{Q}\cdot\hat{e}_y}_{3Qa^2\hat{e}_x}\Big) = -3Qa^2\,\hat{e}'_y$$

$$\longleftrightarrow 3Qa^2\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix} = -3Qa^2\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

③ Obvious

(c)



$\hat{e}'_x = \frac{1}{\sqrt{2}}(\hat{e}_x + \hat{e}_y)$ $\hat{e}_x = \frac{1}{\sqrt{2}}(\hat{e}'_x - \hat{e}'_y)$

$\hat{e}'_y = \frac{1}{\sqrt{2}}(-\hat{e}_x + \hat{e}_y)$ $\hat{e}_y = \frac{1}{\sqrt{2}}(\hat{e}'_x + \hat{e}'_y)$

Method 1: $\overleftrightarrow{Q} = 3Qa^2 \underbrace{\left(\hat{e}_x \otimes \hat{e}_y + \hat{e}_y \otimes \hat{e}_x \right)}$

$$= \frac{1}{2}\left[(\hat{e}'_x - \hat{e}'_y) \otimes (\hat{e}'_x + \hat{e}'_y) + (\hat{e}'_x + \hat{e}'_y) \otimes (\hat{e}'_x - \hat{e}'_y) \right]$$

$$= \hat{e}'_x \otimes \hat{e}'_x - \hat{e}'_y \otimes \hat{e}'_y$$

$$\overleftrightarrow{Q} = 3Qa^2 \left(\hat{e}'_x \otimes \hat{e}'_x - \hat{e}'_y \otimes \hat{e}'_y \right)$$

Method 2: $Q'_{jk} = \hat{e}'_j \cdot \overleftrightarrow{Q} \cdot \hat{e}'_k$

$$= \sum_{\ell, m} \underbrace{\hat{e}'_j \cdot \hat{e}_\ell}_{M_{j\ell}} Q_{\ell m} \underbrace{\hat{e}_m \cdot \hat{e}'_k}_{M^T_{mk}}$$

$$= \sum_{\ell, m} M_{j\ell} Q_{\ell m} M^T_{mk}$$

$$M = \begin{pmatrix} \hat{e}'_x \cdot \hat{e}_x & \hat{e}'_x \cdot \hat{e}_y & \hat{e}'_x \cdot \hat{e}_z \\ \hat{e}'_y \cdot \hat{e}_x & \hat{e}'_y \cdot \hat{e}_y & \hat{e}'_y \cdot \hat{e}_z \\ \hat{e}'_z \cdot \hat{e}_x & \hat{e}'_z \cdot \hat{e}_y & \hat{e}'_z \cdot \hat{e}_z \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} Q'_{xx} & Q'_{xy} & Q'_{xz} \\ Q'_{yx} & Q'_{yy} & Q'_{yz} \\ Q'_{zx} & Q'_{zy} & Q'_{zz} \end{pmatrix} = 3Qa^2 \underbrace{\overbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{M} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \overbrace{\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{M^T}}_{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}}}_{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$\begin{pmatrix} Q'_{xx} & Q'_{xy} & Q'_{xz} \\ Q'_{yx} & Q'_{yy} & Q'_{yz} \\ Q'_{zx} & Q'_{zy} & Q'_{zz} \end{pmatrix} = 3Qa^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$