Phys 366 Mathematical Methods of Physics Fall 2015

Quiz 5 2015 November 19

(100 points)

Problem 1 (100 points) The general solution of the one-dimensional wave equation,

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0 ,$$

is a superposition of a wave going to the right and a wave going to the left. A particular solution is specified by giving the initial waveform, $f(x, 0)$, and the first time derivative at $t = 0$,

$$\left.\frac{\partial f(x,t)}{\partial t}\right|_{(x,0)} .$$

In this problem we work with a wave that is propagating to the right. Such a wave is just the initial waveform propagated to the right, i.e., in an equation,

$$f(x, t) = f(x - vt, 0) .$$

(a) (15 points) *Show* that for a wave going to the right, the first time derivative of f is given by

$$\frac{\partial f(x,t)}{\partial t} = -v\frac{\partial f(x,t)}{\partial x} .$$

We now introduce the Fourier transform $\tilde{f}(k, t)$ with respect to the spatial variable,

$$f(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{f}(k,t)e^{ikx} , \qquad \tilde{f}(k,t) = \int_{-\infty}^{\infty} dx\, f(x,t)e^{-ikx} .$$

We already know that $\tilde{f}(k, t)$ satisfies the ordinary differential equation (k is just a constant as we consider this equation)

$$\frac{d^2\tilde{f}(k,t)}{dt^2} + v^2k^2\tilde{f}(k,t) = 0 .$$

(b) (30 points) *Show* that the condition of part (a) is equivalent to

$$\frac{\partial \tilde{f}(k,t)}{\partial t} = -ivk\tilde{f}(k,t) .$$

You will need to assume that $f(x, t)$ goes to zero at $x = \pm\infty$, but this is not much of an assumption, since it is required to have a meaningful Fourier transform.

(c) (30 points) *Find* the general solution of differential equation for $\tilde{f}(k, t)$ in terms of initial values at $t = 0$, given our assumption of a right-going wave.

(d) (15 points) *Show* that your solution in part (c) corresponds to a wave of arbitrary shape going to the right.

Problem 1. The assumption is that we have a wave going to the right, for which

$$f(x,t) = f(x - vt, 0) \ .$$

(a) The key here is that since the dependence on x and t appears only in the combination $x - vt$, we can relate a derivative with respect to t to a derivative with respect to x:

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial f(x-vt,0)}{\partial t} = -v\frac{\partial f(x-vt,0)}{\partial x} = -v\frac{\partial f(x,t)}{\partial x} \ .$$

We now consider the partial Fourier transform $\tilde{f}(k,t)$, given by

$$f(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{f}(k,t)e^{ikx} \ , \qquad \tilde{f}(k,t) = \int_{-\infty}^{\infty} dx\, f(x,t)e^{-ikx} \ ;$$

$\tilde{f}(k,t)$ satisfies the ordinary differential equation

$$\frac{d^2\tilde{f}(k,t)}{dt^2} + v^2k^2\tilde{f}(k,t) = 0 \ .$$

(b) Differentiating the formula for the Fourier transform, we get

$$\frac{\partial f(x,t)}{\partial x} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, ik\tilde{f}(k,t)e^{ikx} \ .$$

This is the hopefully familiar result that a spatial derivative, $\partial/\partial x$, is equivalent to multiplying by ik in the Fourier domain. But we can also use the result of part (a) to calculate $\partial f/\partial x$ in another way:

$$\frac{\partial f(x,t)}{\partial x} = -\frac{1}{v}\frac{\partial f(x,t)}{\partial t} = -\frac{1}{v}\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{\partial \tilde{f}(k,t)}{\partial t}e^{ikx}$$

Comparing these two equations, we get that

$$\frac{\partial \tilde{f}(k,t)}{\partial t} = -ivk\tilde{f}(k,t) \ .$$

We can also do this more directly, using an integration by parts:

$$\begin{aligned}
\frac{\partial \tilde{f}(k,t)}{\partial t} &= \int_{-\infty}^{\infty} dx\, \frac{\partial f(x,t)}{\partial t}e^{-ikx} \\
&= -v\int_{-\infty}^{\infty} dx\, \frac{\partial f(x,t)}{\partial x}e^{-ikx} \\
&= -vf(x,t)e^{-ikx}\Big|_{x=-\infty}^{x=\infty} + v\int_{-\infty}^{\infty} dx\, \partial f(x,t)(-ik)e^{-ikx} \\
&= -ivk\tilde{f}(k,t) \ .
\end{aligned}$$

(c) The general solution of the differential equation for $\tilde{f}(k,t)$ is

$$\begin{aligned}
\tilde{f}(k,t) &= \tilde{f}(k,0)\cos(vkt) + \frac{1}{vk}\frac{\partial \tilde{f}(k,t)}{\partial t}\bigg|_{(k,0)} \sin(vkt) \\
&= \tilde{f}(k,0)[\cos(vkt) - i\sin(kvt)] \\
&= \tilde{f}(k,0)e^{-ivkt} \ .
\end{aligned}$$

(d)

$$f(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{f}(k,t)e^{ikx} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{f}(k,0)e^{ik(x-vt)} = f(x-vt,0) \ .$$

So we get back to where we started with a wave of arbitrary shape going to the right.