**Phys 366 Mathematical Methods of Physics Fall 2016**

**Quiz 2 2016 October 4**
**(100 points)**

**Problem 1 (40 points)** A particle moves along a circular path given in cylindrical coördinates by

$$\rho(t) = a \qquad \text{and} \qquad \phi(t) = \pi(1 - e^{-\gamma t}) \qquad \text{and} \qquad z(t) = 0\ .$$

(a) (10 points) *Give* the particle's position vector $\mathbf{r}(t)$.

(b) (10 points) *Give* the velocity $\mathbf{v}(t)$ and the speed $v(t) = |\,\mathbf{v}(t)|$.

(c) (10 points) *Give* the acceleration $\mathbf{a}(t)$.

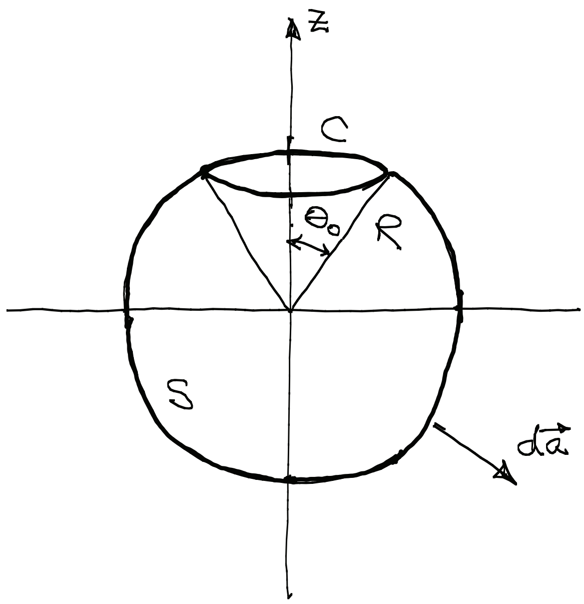
(d) (10 points) *Draw* the entire trajectory for $t \geq 0$, and on your drawing, *draw* and *label* the position vector, velocity, and acceleration at time $t = 0$.

**Problem 2 (60 points)** Let $\mathbf{A}$ be the vector field

$$\mathbf{A} = \hat{\mathbf{e}}_r L r (3\cos^2\theta - 1) + \hat{\mathbf{e}}_\theta M r \sin\theta\cos\theta + \hat{\mathbf{e}}_\phi N r \sin\theta\ .$$

This vector field is given in spherical coördinates, and $L$, $M$, and $N$ are constants.

Consider a spherical shell of radius $R$, which is centered at the origin of a spherical coordinate system. Let $S$ be the part of the shell defined by $\theta \geq \theta_0$. The surface $S$, with outward-directed area element $d\mathbf{a}$, is shown in the drawing below. Notice that $S$ is shaped like a goldfish bowl. The boundary of $S$ is the curve $C$, which is the lip of the goldfish bowl.



(a) (20 points) *Evaluate* the surface integral

$$\int_S \nabla \times \mathbf{A} \cdot d\mathbf{a}$$

directly in terms of $L$, $M$, $N$, $R$, and $\theta_0$.

(b) (20 points) Use Stokes's theorem to convert the surface integral into a line integral along $C$. *Explain* your choice for the direction of the integration along $C$. *Show* that the line integral gives the same result as in part (a).

(c) (20 points) Use Stokes's theorem to convert the original surface integral into a surface integral over the yarmulke (kippah or skullcap) that caps the spherical shell, i.e., the part $Y$ of the spherical shell that is defined by $\theta \leq \theta_0$. *Show* that the integral over $Y$ gives the same result as in part (a).

## Derivatives of basis vectors along a trajectory

Cartesian coördinates

$$\frac{d\hat{\mathbf{x}}}{dt} = 0 \; , \qquad \frac{d\hat{\mathbf{y}}}{dt} = 0 \; , \qquad \frac{d\hat{\mathbf{z}}}{dt} = 0$$

Cylindrical coördinates

$$\frac{d\hat{\boldsymbol{\rho}}}{dt} = \dot{\phi}\hat{\boldsymbol{\phi}} \; , \qquad \frac{d\hat{\boldsymbol{\phi}}}{dt} = -\dot{\phi}\hat{\boldsymbol{\rho}} \; , \qquad \frac{d\hat{\mathbf{z}}}{dt} = 0$$

Spherical coördinates

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{\phi}\sin\theta\,\hat{\boldsymbol{\phi}} \; , \qquad \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\theta}\,\hat{\mathbf{r}} + \dot{\phi}\cos\theta\,\hat{\boldsymbol{\phi}} \; , \qquad \frac{d\hat{\boldsymbol{\phi}}}{dt} = -\dot{\phi}\sin\theta\,\hat{\mathbf{r}} - \dot{\phi}\cos\theta\,\hat{\boldsymbol{\theta}}$$

## Formulae for the curl

Cartesian coördinates

$$\nabla \times \mathbf{F} = \sum_{k,l} \hat{\mathbf{e}}_j \epsilon_{jkl} \frac{\partial F_l}{\partial x_k} = \hat{\mathbf{e}}_x \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{\mathbf{e}}_y \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{\mathbf{e}}_z \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

Cylindrical coördinates

$$\nabla \times \mathbf{F} = \hat{\mathbf{e}}_\rho \left( \frac{1}{\rho} \frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z} \right) + \hat{\mathbf{e}}_\phi \left( \frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho} \right) + \hat{\mathbf{e}}_z \frac{1}{\rho} \left( \frac{\partial (\rho F_\phi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \phi} \right)$$

Spherical coördinates

$$\begin{aligned} \nabla \times \mathbf{F} = {} & \hat{\mathbf{e}}_r \frac{1}{r\sin\theta} \left( \frac{\partial (\sin\theta\, F_\phi)}{\partial \theta} - \frac{\partial F_\theta}{\partial \phi} \right) \\ & + \hat{\mathbf{e}}_\theta \frac{1}{r} \left( \frac{1}{\sin\theta} \frac{\partial F_r}{\partial \phi} - \frac{\partial (r F_\phi)}{\partial r} \right) + \hat{\mathbf{e}}_\phi \frac{1}{r} \left( \frac{\partial (r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} \right) \end{aligned}$$