Problem 1.

(a)

$$\begin{aligned}
\tilde{s}(\omega) &= \int_{-\infty}^{\infty} dt\, s(t) e^{i\omega t} \\
&= \int_{0}^{\infty} dt\, e^{(-\beta+i\omega)t} \\
&= \frac{e^{(-\beta+i\omega)t}}{-\beta+i\omega}\bigg|_{t=0}^{t=\infty} \\
&= \frac{1}{\beta - i\omega}
\end{aligned}$$

(b)

$$\beta v(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \beta\tilde{v}(\omega) e^{-i\omega t}$$

The Fourier transform of $\beta v(t)$ is $\beta\tilde{v}(\omega)$.

$$\dot{v}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, [-i\omega\tilde{v}(\omega)] e^{-i\omega t}$$

The Fourier transform of $\dot{v}(t)$ is $-i\omega\tilde{v}(\omega)$. This is the $d/dt \to -i\omega$ rule for translating a temporal derivative to the temporal Fourier domain.

$$\frac{f(t)}{m} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \frac{\tilde{f}(\omega)}{m} e^{-i\omega t}$$

The Fourier transform of $f(t)/m$ is $\tilde{f}(\omega)/m$.

Equating Fourier transforms on the two sides of Eq. (1) gives

$$-i\omega\tilde{v}(\omega) + \beta\tilde{v}(\omega) = \frac{\tilde{f}(\omega)}{m} \qquad \Longrightarrow \qquad \tilde{v}(\omega) = \frac{1}{m}\frac{\tilde{f}(\omega)}{\beta - i\omega} .$$

(c) One way to proceed is to notice that

$$\tilde{v}(\omega) = \frac{\tilde{f}(\omega)}{m}\tilde{s}(\omega) ,$$

and then to invoke the convolution theorem to write

$$v(t) = \int_{-\infty}^{\infty} dt'\, \frac{f(t')}{m} s(t-t') = \int_{-\infty}^{\infty} dt'\, \frac{f(t')}{m} H(t-t') e^{-\beta(t-t')} = \int_{-\infty}^{t} dt'\, \underbrace{\frac{e^{-\beta(t-t')}}{m}}_{=\,g(t-t')} f(t') .$$

One can also proceed directly, basically deriving the convolution theorem along the way:

$$\begin{aligned}
v(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \tilde{v}(\omega) e^{-i\omega t} \\
&= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t} \frac{1}{m}\frac{1}{\beta - i\omega} \tilde{f}(\omega) \\
&= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t} \frac{1}{m}\frac{1}{\beta - i\omega} \int_{-\infty}^{\infty} dt'\, f(t') e^{i\omega t'} \\
&= \int_{-\infty}^{\infty} dt'\, f(t') \underbrace{\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \frac{1}{m}\frac{1}{\beta - i\omega} e^{-i\omega(t-t')}}_{=\,\frac{s(t-t')}{m}\,=\,\frac{H(t-t')e^{-\beta(t-t')}}{m}} \\
&= \int_{-\infty}^{t} dt'\, \underbrace{\frac{e^{-\beta(t-t')}}{m}}_{=\,g(t-t')} f(t') .
\end{aligned}$$

(d) When one differentiates $v(t)$, one has to take the derivative with respect to the upper integration limit, which means evaluating the integrand at $t' = t$, and also to take the derivative with respect to the t inside the integral:

$$\dot{v}(t) = \frac{f(t)}{m} - \beta \underbrace{\int_{-\infty}^{t} dt' \, \frac{e^{-\beta(t-t')}}{m} f(t')}_{= v(t)} = \frac{f(t)}{m} - \beta v(t) \; .$$