Phys 406 Electricity and Magnetism II Fall 2008

Homework Assignment #7 Due Monday, October 13
(30 points) (at lecture)

7.1 (10 points) **Birefringence.** Consider a linear, homogeneous, nonconducting, but anisotropic dielectric medium in which

$$D_x = \epsilon_x E_x \ , \qquad D_y = \epsilon_y E_y \ , \qquad D_z = \epsilon_z E_z \ .$$

Such a medium, called *birefringent*, is used to make quarter- and half-wave plates.

In this problem we are interested in monochromatic plane waves with frequency ω that propagate in the $+x$ direction. Throughout this problem we use phasors, remembering that the actual electric and magnetic fields are given by the real parts of the phasors. You do not have to take the real part unless you want to.

(a) A wave that is linearly polarized along the y axis, i.e., a wave with electric field

$$\mathbf{E} = \hat{\mathbf{e}}_y E_0 e^{i(kx-\omega t)} \ ,$$

is a solution of the Maxwell equations. *Derive* the associated magnetic field **B** and the phase velocity v_y of this wave.

(b) A wave that is linearly polarized along the z axis, i.e., a wave with electric field

$$\mathbf{E} = \hat{\mathbf{e}}_z E_0 e^{i(kx-\omega t)} \ ,$$

is a solution of the Maxwell equations. *Give* the associated magnetic field **B** and the phase velocity v_z of this wave. Don't do any work on this part if you can simply translate your results for part (a).

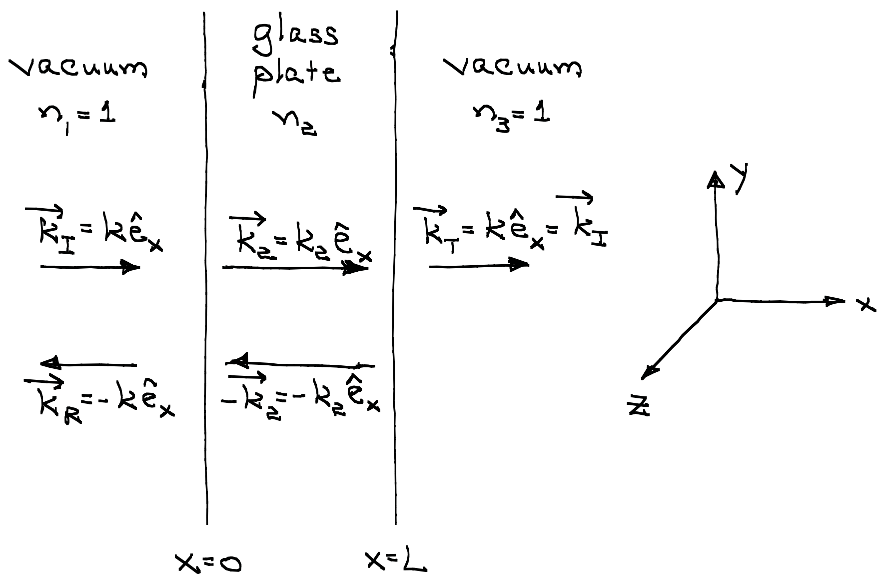
(c) Suppose that $x = 0$ is the left boundary of the dielectric medium: the medium occupies the region $x > 0$, and the region $x < 0$ is vacuum. A wave incident on the medium from the vacuum is transmitted into the medium. Assume that at $x = 0$ the wave propagating into the medium is linearly polarized along the unit vector

$$\frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_y + \hat{\mathbf{e}}_z) \ .$$

Write an expression for the electric field **E** at all $x > 0$. (Hint: Use the superposition principle.)

(d) Now assume that $v_y > v_z$. (i) *Find* the smallest value of x in the medium such that the wave has right-circular polarization. (ii) *Find* the smallest value of x in the medium such that the wave is linearly polarized along the unit vector $(\hat{\mathbf{e}}_y - \hat{\mathbf{e}}_z)/\sqrt{2}$ (i.e., orthogonal to the linear polarization at $x = 0$). (iii) *Find* the smallest value of x in the medium such that the wave has left-circular polarization. If the medium has a right boundary at the position found in (i), it is called a *quarter-wave plate*; if the medium has a right boundary at the position found in (ii), it is called a *half-wave plate*. Can you explain the origin of these terms?

7.2 (10 points) **Fabry-Perot cavity.** Consider a plate of glass of thickness L, as shown in the drawing below. The glass has electric permittivity $\epsilon_2 \equiv \epsilon$, magnetic permeability $\mu_2 = \mu_0$, and index of refraction $n_2 = \sqrt{\epsilon_2/\epsilon_0} \equiv n$. Within the glass plate, electromagnetic waves have a phase velocity $v_2 = c/n \equiv v$.



A plane electromagnetic wave, linearly polarized in the y direction, is incident normally on the glass plate from the vacuum region on the left; the electric and magnetic fields of the incident wave ($x < 0$) are given by

$$\mathbf{E}_I = E_I \hat{\mathbf{e}}_y e^{i(kx-\omega t)} \ , \qquad \mathbf{B}_I = \frac{1}{c} E_I \hat{\mathbf{e}}_z e^{i(kx-\omega t)} \ ,$$

where $\omega = ck$ and E_I is a real amplitude.

Some of the incident wave is reflected, and some is transmitted through to the vacuum region on the right side of the glass plate. The electric and magnetic fields of the reflected wave ($x < 0$) are given by

$$\mathbf{E}_R = \tilde{E}_R \hat{\mathbf{e}}_y e^{-i(kx+\omega t)} \ , \qquad \mathbf{B}_R = -\frac{1}{c} \tilde{E}_R \hat{\mathbf{e}}_z e^{-i(kx+\omega t)} \ ,$$

and the electric and magnetic fields of the transmitted wave ($x > L$) are given by

$$\mathbf{E}_T = \tilde{E}_T \hat{\mathbf{e}}_y e^{i(kx-\omega t)} \ , \qquad \mathbf{B}_T = \frac{1}{c} \tilde{E}_T \hat{\mathbf{e}}_z e^{i(kx-\omega t)} \ .$$

Inside the glass plate there are both left-going and right-going waves. The electric and magnetic fields of the right-going wave ($0 < x < L$) are given by

$$\mathbf{E}_2^{(R)} = \tilde{C} \hat{\mathbf{e}}_y e^{i(k_2 x-\omega t)} \ , \qquad \mathbf{B}_2^{(R)} = \frac{1}{v} \tilde{C} \hat{\mathbf{e}}_z e^{i(k_2 x-\omega t)} \ ,$$

where $\omega = vk$, and the electric and magnetic fields of the left-going wave ($0 < x < L$) are given by

$$\mathbf{E}_2^{(L)} = \tilde{D}\hat{\mathbf{e}}_y e^{-i(k_2 x + \omega t)} \;, \qquad \mathbf{B}_2^{(L)} = -\frac{1}{v}\tilde{D}\hat{\mathbf{e}}_z e^{-i(k_2 x + \omega t)} \;.$$

(a) The boundary conditions at the left side of the glass plate ($x = 0$) give two relations among E_I, $\tilde{E}_R$, $\tilde{C}$, and $\tilde{D}$. *Find* these two relations.

(b) The boundary conditions at the right side of the glass plate ($x = L$) give two relations among $\tilde{E}_T$, $\tilde{C}$, and $\tilde{D}$. *Find* these two relations.

(c) *Show* that when L is an integral number of half-wavelengths (within the glass), all of the wave is transmitted, and none is reflected.

(d) *Show* that when L is an integral number of half-wavelengths plus a quarter-wavelength (within the glass), the reflected and transmitted complex amplitudes are given by

$$\tilde{E}_R = \frac{1 - n^2}{1 + n^2} E_I \;, \qquad \tilde{E}_T = \frac{2n}{1 + n^2} E_I e^{-i(k - k_2)L} \;.$$

(e) For an arbitrary width of the plate, *find* the ratios $\tilde{E}_R/E_I$ and $\tilde{E}_T/E_I$. *Find* the reflection and transmission coefficients, R and T, i.e., the ratios of reflected and transmitted intensities to incident intensity, and *show* that energy is conserved. In doing the algebra for this problem, you might it useful to introduce $\tilde{F}_T \equiv \tilde{E}_T e^{i(k-k_2)L}$; $\tilde{F}_T$ is just a different complex amplitude for the transmitted wave, which differs from $\tilde{E}_T$ by a wavelength-dependent phase.

7.3 (10 points) Challenge problem.