This is an examination for Phys 503. Your score will determine 20% of your grade for the course.

This is a take-home, open-book exam. You may use the textbook, your own notes, your own homework assignments, all class handouts, including solution sets for homework assignments, but do not consult books or any other material, including online material, other than our primary textbook.

You may use as much time as you wish to complete the exam, as long as it is turned in by the deadline. Your completed exam must be solely your own work; do not consult anyone else in doing the exam.

Do your work on sheets of paper separate from the exam. The solution you hand in for a problem should be neat and legible and in a logical order. It should represent your best effort at solving the problem and should not include false starts and detours that led nowhere.

Staple your solution(s) to this page, and label this page with your *printed* name. Then *sign and date the pledge below,* and turn in the exam as instructed. You may keep the exam question(s).

I have obeyed the rules in taking this exam. In particular, I have not consulted anyone about the exam, and the completed exam is solely my own work.

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Signature   Date

Problem 1 (20 points) The drawing below shows a bead of mass m that slides without friction on a wire whose shape is described by a parabola $z = r^2/d$, where r is the distance from the z axis. The wire rotates with angular velocity Ω about the z axis.
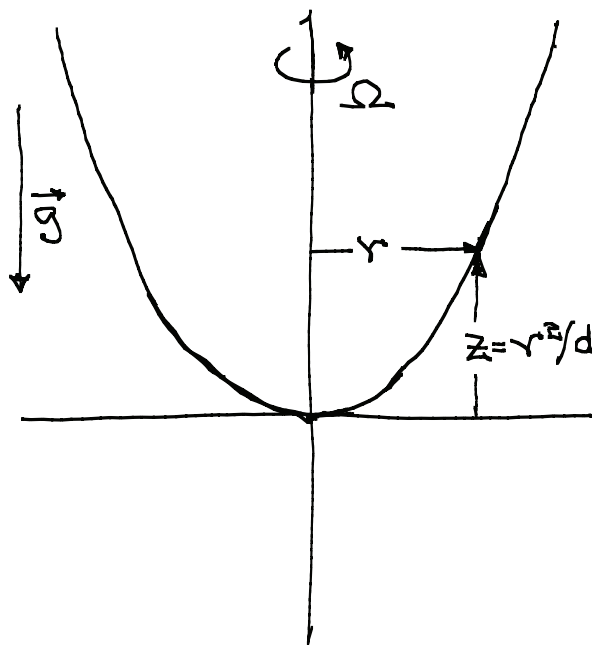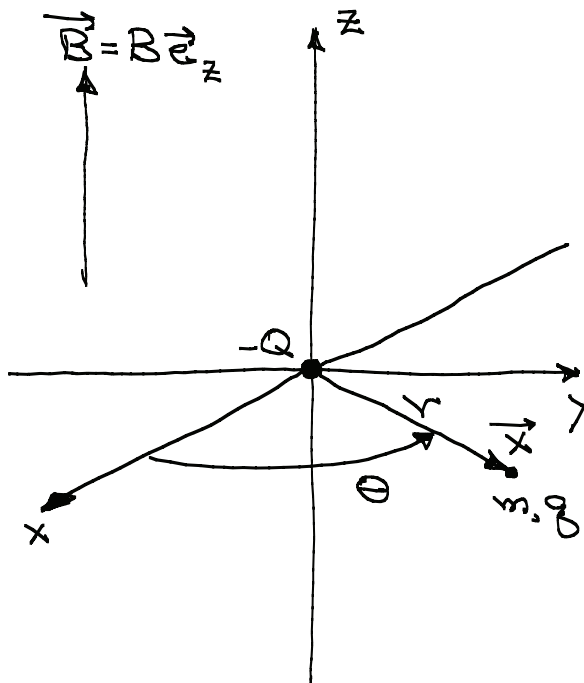

(a) (6 points) Find the Lagrangian L in terms of the generalized coördinate r.

(b) (6 points) Find the Hamiltonian H in terms of the coördinate r and its conjugate momentum p_r.

(c) (8 points) For what range of values of Ω is $r = 0$ a *stable* equilibrium position? Within that range of values, what is the angular frequency ω of small oscillations about $r = 0$? (Hint: You should be able to answer this part just by looking at the Lagrangian or Hamiltonian, without writing out the equations of motion.)

Problem 2 (40 points) A particle of mass m and charge q moves in the electric field produced by a point charge $-Q$ at the origin and in a uniform magnetic field $\mathbf{B} = B\mathbf{e}_z$ that points in the z direction. The scalar and vector potentials for these fields (cgs Gaussian) are

$$\phi = -\frac{Q}{r} \qquad \text{and} \qquad \mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{x} \,.$$

Assume that the particle's motion is restricted to the x-y plane. Describe the motion in terms of usual radial and angular coördinates, r and θ. It will be useful to introduce a frequency

$$\Omega = qB/2mc \,.$$

(a) (10 points) From the Lagrangian

$$L = \frac{1}{2}m|\dot{\mathbf{x}}|^2 - q\phi + \frac{q}{c}\dot{\mathbf{x}} \cdot \mathbf{A} \;,$$

derive the particle's Hamiltonian H in terms of coördinates, r and θ, and canonical momenta, p_r and p_θ.

(b) (10 points) Show that the coördinate transformation

$$\phi = \theta + \Omega t \qquad \text{and} \qquad p_\phi = p_\theta$$

is a canonical transformation, and find the transformed Hamiltonian in terms of the new coördinates.

(c) (4 points) Give two independent conserved quantities for the particle's motion.

(d) (16 points) Identify an effective potential $V(r)$ for the radial motion, and then find the temporal frequency ω of small radial oscillations about a circular orbit in terms of m, q, Q, Ω, and the circular-orbit radius r_0. Notice that you aren't required to solve for the circular-orbit radius in terms of other parameters. [Hint: You shouldn't need anything except the transformed Hamiltonian of part (c).]

Problem 3 (40 points) A particle that has coördinate-dependent mass mq^2 is attached to a spring that has spring constant $k = mg$. The particle's Hamiltonian is

$$H = \frac{p^2}{2mq^2} + \frac{1}{2}mgq^2 \;.$$

(a) (10 points) Derive the Hamilton equations of motion for q and p.

(b) (15 points) Show that the coördinate transformation

$$z = \frac{1}{2}q^2 \qquad \text{and} \qquad p_z = \frac{p}{q}$$

is a canonical transformation. Derive the Hamilton equations of motion for z and p_z, and find the general solutions for $z(t)$ and $p_z(t)$ in terms of the initial values z_0 and p_{z0}.

(c) (15 points) Let

$$P = H = \frac{p_z^2}{2m} + mgz .$$

Find a coördinate Q such that Q and P make up a pair of canonical coördinates. Derive the Hamilton equations of motion for Q and P, and find the general solutions for $Q(t)$ and $P(t)$ in terms of the initial values Q_0 and P_0.