2.7 (10 points) Challenge problem. A particle of mass m slides without friction on a cylindrically symmetric funnel defined by $z = -a^2/r$.
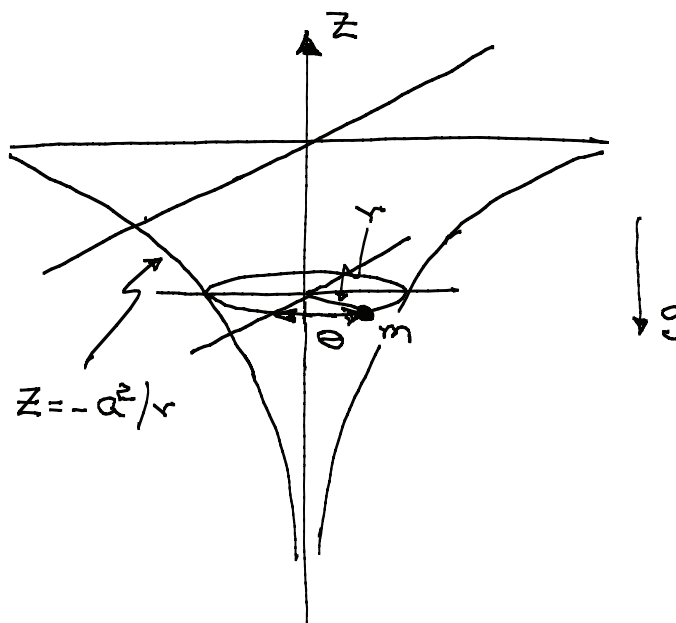
(a) Give a Lagrangian in polar coördinates r and θ. In what limit does this system mimic an attractive $1/r^2$ central force?

(b) Derive the equation of motion for r.

(c) What is the radius r_0 of a circular orbit with angular momentum l?

(d) Derive an equation for the shape of an orbit in terms of $u \equiv 1/r$.

(e) Apply the equation of part (d) to a nearly circular orbit to determine the precession of the point of minimum r.