Normal coördinates and quantum mechanics
CMC
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Consider a quantum system with Hamiltonian H. The stationary states are the eigenstates of H: $H|E\rangle = E|E\rangle$. The state vector $|\psi(t)\rangle$ can be written in terms of an arbitrary basis, $\{|j\rangle\}$, as

$$|\psi(t)\rangle = \sum_j |j\rangle\langle j|\psi(t)\rangle = \sum_j \eta_j(t)|j\rangle \,,$$

where the "original coördinates" are the amplitudes $\eta_j(t) = \langle j|\psi(t)\rangle$. The state vector can also be written in terms of the energy eigenbasis as

$$|\psi(t)\rangle = \sum_E |E\rangle\langle E|\psi(t)\rangle = \sum_E \zeta_E(0)e^{-iEt/\hbar}|E\rangle \,,$$

where the amplitudes

$$\zeta_E(t) = \langle E|\psi(t)\rangle = \langle E|e^{-iHt/\hbar}|\psi(0)\rangle = e^{-iEt/\hbar}\langle E|\psi(0)\rangle = e^{-iEt/\hbar}\zeta_E(0)$$

are the normal coördinates because they oscillate sinusoidally in time.

The original coördinates are related to the normal coördinates by

$$\eta_j(t) = \langle j|\psi(t)\rangle = \sum_E \underbrace{\langle j|E\rangle}_{=\,a_{jE}}\langle E|\psi(t)\rangle = \sum_E a_{jE}\zeta_E(0)e^{-iEt/\hbar} \,.$$

The energy eigenstates are the normal modes; they are related to the basis $|j\rangle$ by

$$|E\rangle = \sum_j |j\rangle\langle j|E\rangle = a_{jE}|j\rangle \,.$$

The matrix elements a_{jE} give the "shape" of normal mode E in the original basis.

The only difference with what we are doing in linear mechanical systems is that the variables are real instead of complex and thus obey second-order, linear differential equations in time, instead of the first-order, linear Schrödinger equation. Once one is past that difference, the whole thing is identical. If we were to master the bra-ket notation for the real vector spaces in mechanics, the question of which order to write the indices in and whether to use a transformation matrix or its transpose would disappear as questions, since the bra-ket notation takes care of all that without any need for thinking.