Phys 521 Quantum Mechanics I **Fall 2011**

Exam #3 **Due Monday, November 21**
(16% of course grade) **(at noon in TA's mailbox)**

This is an examination for Phys 521. Your score will determine 16% of your grade for the course.

This is a take-home, open-book exam. You may use the textbook, your own notes, your own homework assignments, all class handouts, including solution sets for homework assignments, but do not consult books other than our primary textbook. You may use as much time as you want to complete the exam, as long as it is turned in by the deadline. Your completed exam must be solely your own work; do not consult anyone else in doing the exam.

Do your work on sheets of paper separate from the exam. The solution you hand in for a problem should be neat and legible and in a logical order. It should represent your best effort at solving the problem and should not include false starts and detours that led nowhere.

Staple your solution(s) to this page, and label this page with your *printed* name. Then *sign and date the pledge below,* and turn in the exam as instructed. You may keep the exam question(s).

I have obeyed the rules in taking this exam. In particular, I have not consulted anyone about the exam, and the completed exam is solely my own work.

____________________________________ __________________

Signature Date

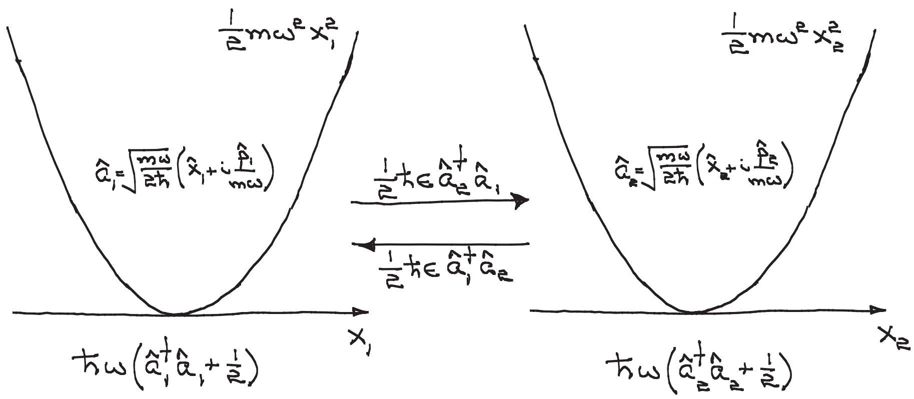
Phys 521 **Quantum Mechanics I** **Fall 2011**

Exam #3 **(100 points)**

Problem 1 (100 points) Consider two identical harmonic oscillators, both with mass m and and resonance frequency ω. The Hamiltonian of the two oscillators is

$$\hat{H}_0 = \frac{\hat{p}_1^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_1^2 + \frac{\hat{p}_2^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_2^2 \ .$$

The position and momentum operators obey canonical commutation relations, $[\hat{x}_j, \hat{p}_k] = i\hbar\delta_{jk}$ for $j, k = 1, 2$. We will call $\hat{H}_0$ the "bare" Hamiltonian because it is bare of the coupling we introduce below, and we will call the operators associated with $\hat{H}_0$ bare operators. The situation is depicted in the diagram below.



If we introduce bare creation and annihilation operators for the two oscillators,

$$\hat{a}_j = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_j + i\frac{\hat{p}_j}{m\omega}\right) \ , \qquad \hat{a}_j^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_j - i\frac{\hat{p}_j}{m\omega}\right) \ , \qquad j = 1, 2,$$

the bare Hamiltonian takes the form

$$\hat{H}_0 = \hbar\omega(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 + 1) \ .$$

The eigenstates of $\hat{H}_0$ are the states with n_1 quanta in the first oscillator and n_2 quanta in the second:

$$\overline{|n_1, n_2\rangle} = \frac{(\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2}}{\sqrt{n_1!\,n_2!}}\overline{|0, 0\rangle} \ ,$$

where $\overline{|0, 0\rangle}$ is the vacuum state, i.e., the state with no quanta in either oscillator. The corresponding eigenvalue of $\hat{H}_0$ is $\hbar\omega(n_1 + n_2 + 1)$. We use an overbar to denote states associated with the bare operators, and we call the states $\overline{|n_1, n_2\rangle}$ the bare number states.

Now we suppose that the two oscillators are coupled in such a way that quanta can tunnel between the two oscillators. Such a coupling is described by an interaction Hamiltonian $\frac{1}{2}\hbar\epsilon(\hat{a}_2^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_2)$, where ϵ characterizes the strength of the tunneling in frequency units. The term $\hat{a}_2^\dagger \hat{a}_1$ describes destruction of a quantum in the first oscillator and creation of a quantum in the second oscillator and thus corresponds to tunneling from the first oscillator to the second. The term $\hat{a}_1^\dagger \hat{a}_2$ describes destruction of a quantum in the second oscillator and creation of a quantum in the first oscillator and thus corresponds to tunneling from the second oscillator to the first. The total Hamiltonian is now given by

$$\hat{H} = \hat{H}_0 + \frac{1}{2}\hbar\epsilon(\hat{a}_2^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2)\ .$$

We will assume the tunneling interaction is weak, which is to say that $\epsilon \ll \omega$.

You will recall from a course in classical mechanics that it is useful to do a canonical transformation to coördinates that describe uncoupled oscillators. In the situation we are considering, this transformation is very simple. We introduce sum and difference annihilation operators

$$\hat{a}_\pm = \frac{1}{\sqrt{2}}(\hat{a}_2 \pm \hat{a}_1)\ ,$$

It is easy to check that these new annihilation operators and the corresponding creation operators satisfy canonical commutation relations and that the total Hamiltonian takes the form of uncoupled sum and difference oscillators, with frequencies $\omega_\pm = \omega \pm \epsilon/2$:

$$\hat{H} = \hbar\omega_+\left(\hat{a}_+^\dagger \hat{a}_+ + \frac{1}{2}\right) + \hbar\omega_-\left(\hat{a}_-^\dagger \hat{a}_- + \frac{1}{2}\right)\ .$$

We denote the energy eigenstates of the total Hamiltonian by $|n_+, n_-\rangle$, where n_+ and n_- are the numbers of quanta in the sum and difference oscillators.

(a) (20 points) *Give* the lowest six of the energy eigenstates $|n_+, n_-\rangle$ in terms of the bare number states $\overline{|n_1, n_2\rangle}$, and *give* the corresponding energy eigenvalues. (It should be obvious how to do this for all the energy eigenstates, but I'm not asking you to do that here, because it becomes a question of efficient notation.)

(b) (20 points) *Derive* and *solve* the Heisenberg-picture equations of motion for the sum and difference annihilation operators and for the bare annihilation operators. For both types of annihilation operators, solve in terms of initial conditions at $t = 0$.

(c) (20 points) Suppose the state of the two oscillators at time $t = 0$ is a coherent state with $\langle \hat{x}_2 \rangle_0 = A$, where A is real and positive, and $\langle \hat{x}_1 \rangle_0 = 0 = \langle \hat{p}_1 \rangle_0 = \langle \hat{p}_2 \rangle_0$. This is the quantum version of pulling the second oscillator out to position A and releasing it from rest. *Give* the expectation values and uncertainties of the position and momentum operators at time t.

For the remainder of the problem, we will assume that the two oscillators have one quantum of excitation. From your work in part (b), you should know that

$$|1,0\rangle = \frac{1}{\sqrt{2}}\big(\overline{|0,1\rangle} + \overline{|1,0\rangle}\big) \equiv |+\rangle\ , \qquad |0,1\rangle = \frac{1}{\sqrt{2}}\big(\overline{|0,1\rangle} - \overline{|1,0\rangle}\big) \equiv |-\rangle\ .$$

[If you didn't find this in part (b), perhaps you should check your work for that part.] The one-quantum subspace is a two-dimensional subspace, decoupled from the rest of the oscillators' Hilbert space, so we can treat it like any other two-dimensional subspace. We use the shorthand notation, $|\pm\rangle$, for basis vectors in this subspace and introduce the standard Pauli operators based on this shorthand:

$$\hat{\sigma}_z = |+\rangle\langle+| - |-\rangle\langle-| \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

$$\hat{\sigma}_x = |+\rangle\langle-| + |-\rangle\langle+| \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

$$\hat{\sigma}_y = -i|+\rangle\langle-| + i|-\rangle\langle+| \longleftrightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

(d) (15 points) *Give* the eigenstates of the three Pauli operators in terms of the energy eigenstates, $|1,0\rangle$ and $|0,1\rangle$, and in terms of the bare number states, $\overline{|1,0\rangle}$ and $\overline{|0,1\rangle}$.

(e) (10 points) *Give* the total Hamiltonian $\hat{H}$ (restricted to the one-quantum subspace) in terms of the Pauli operators.

(f) (15 points) Suppose the initial state of the two oscillators is the state with one quantum of excitation in the second oscillator, i.e., the state $\overline{|0,1\rangle}$. This is the one-quantum analogue of the initial state in part (c). *Find* the state $|\psi(t)\rangle$ at time t in terms of the energy eigenstates, $|1,0\rangle$ and $|0,1\rangle$, and in terms of the bare number states, $\overline{|0,1\rangle}$ and $\overline{|1,0\rangle}$.