Phys 521 Quantum Mechanics I **Fall 2011**

Homework Assignment #3 **Due Tuesday, October 4**
(70 points) **(at lecture)**

3.7 (10 points) Challenge problem (b). **Commutator algebra.**

(a) If $\hat{A}$ and $\hat{B}$ both commute with their commutator, $[\hat{A}, \hat{B}]$, *prove* that

$$[\hat{A}, \hat{B}^n] = n\hat{B}^{n-1}[\hat{A}, \hat{B}],$$

$$[\hat{A}^n, \hat{B}] = n\hat{A}^{n-1}[\hat{A}, \hat{B}].$$

(Hint: Proceed by induction.)

(b) *Prove* that

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \frac{1}{3!}[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \cdots.$$

[Hint: Define the operator function $\hat{f}(\lambda) \equiv e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}$, derive its Taylor expansion about $\lambda = 0$, and then set $\lambda = 1$.]

(c) If $[\hat{A}, \hat{B}] = \gamma\hat{B}$, where γ is a complex number, *show* that

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = e^{\gamma}\hat{B}.$$

(d) If $\hat{A}$ and $\hat{B}$ both commute with their commutator, $[\hat{A}, \hat{B}]$, *prove* the Baker-Campbell-Hausdorf (BCH) identity,

$$e^{\hat{A}+\hat{B}} = e^{-\frac{1}{2}[\hat{A},\hat{B}]}e^{\hat{A}}e^{\hat{B}}.$$

[Hint: Consider the operator function $\hat{f}(\lambda) \equiv e^{\lambda(\hat{A}+\hat{B})}e^{-\lambda\hat{B}}e^{-\lambda\hat{A}}$, establish the differential equation $d\hat{f}/d\lambda = -\lambda[\hat{A}, \hat{B}]\hat{f}$, and integrate this equation.]