Phys 521 Quantum Mechanics I **Fall 2011**

Homework Assignment #5 **Due Tuesday, November 1**
(40 points) **(at lecture)**

5.1 (10 points) Challenge problem (a)

5.2 (10 points) Consider a harmonic oscillator with mass m, resonant frequency ω, position operator X, and momentum operator P (in this problem, X and P are the usual position and momentum operators, in accordance with C-T's notation, not the dimensionless operators introduced in the lectures). The creation and annihilation operators are given by

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(X + i\frac{P}{m\omega}\right) , \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(X - i\frac{P}{m\omega}\right) .$$

The Hamiltonian is $H = \hbar\omega a^\dagger a$, where the zero-point energy has been omitted, and the unitary evolution operator is

$$U(t,0) = \exp\!\left(-\frac{i}{\hbar}Ht\right) = e^{-i\omega t a^\dagger a} .$$

This problem deals with with the minimum-uncertainty states $|\psi_r\rangle$, which satisfy

$$(a\cosh r + a^\dagger \sinh r)|\psi_r\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left(e^r X + ie^{-r}\frac{P}{m\omega}\right)|\psi_r\rangle = 0 ,$$

where r is a real number called the *squeeze parameter*. Up to an arbitrary phase there is one minimum-uncertainty state for each value of r, as can be verified by calculating the wave function $\langle x|\psi_r\rangle$.

(a) *Evaluate* the following expectation values with respect to the state $|\psi_r\rangle$: $\langle X\rangle$, $\langle P\rangle$, $\langle X^2\rangle$, $\langle P^2\rangle$, and $\langle XP + PX\rangle$. (Hint: If you don't find a minimum-uncertainty product, you made a mistake.)

(b) The unitary *squeeze operator* is defined by

$$S(r,\theta) \equiv \exp\!\left(\frac{1}{2}r\left(a^2 e^{-2i\theta} - (a^\dagger)^2 e^{2i\theta}\right)\right) .$$

Show that

$$S(r,\theta) a S^\dagger(r,\theta) = a\cosh r + a^\dagger e^{2i\theta}\sinh r .$$

(c) Squeezed states are obtained by letting the squeeze operator act on the oscillator's ground state $|\varphi_0\rangle$; that is, the squeezed state $|\phi_{r,\theta}\rangle$ is given by

$$|\phi_{r,\theta}\rangle \equiv S(r,\theta)|\varphi_0\rangle .$$

Show that $|\psi_r\rangle$ can be taken to be the squeezed state $|\phi_{r,0}\rangle$.

(d) Suppose that the oscillator's initial state is $|\psi(0)\rangle = |\phi_{r,0}\rangle$. *Show* that the state $|\psi(t)\rangle$ at an arbitrary time t is a squeezed state $|\phi_{r(t),\theta(t)}\rangle$, and *determine* $r(t)$ and $\theta(t)$ as functions of t. *Show* that at time $t = \pi/2\omega$, i.e., a quarter of the way through an oscillation, the state is a minimum-uncertainty-state with the sign of the squeeze parameter reversed.

(e) *Find* the wave function $\psi_r(x) = \langle x|\psi_r\rangle$ of the minimum-uncertainty state (up to an irrelevant global phase). [Hint: Find a first-order differential equation satisfied by $\psi_r(x)$.]

You might find one or more of the following properties useful.

BCH identity: $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ if A and B commute with $[A,B]$

$$e^B A e^{-B} = A + [B,A] + \frac{1}{2!}\big[B,[B,A]\big] + \frac{1}{3!}\Big[B,\big[B,[B,A]\big]\Big] + \cdots$$

Heisenberg-picture evolution: $U^\dagger(t,0)aU(t,0) = e^{i\omega t a^\dagger a} a e^{-i\omega t a^\dagger a} = a^{-i\omega t}$

Hyperbolic cosine: $\cosh r \equiv \dfrac{1}{2}(e^r + e^{-r}) = \displaystyle\sum_{\text{even } k\geq 0} \frac{r^k}{k!}$

Hyperbolic sine: $\sinh r \equiv \dfrac{1}{2}(e^r - e^{-r}) = \displaystyle\sum_{\text{odd } k\geq 0} \frac{r^k}{k!}$

$\cosh r + \sinh r = e^r \qquad \cosh r - \sinh r = e^{-r} \qquad \cosh^2 r - \sinh^2 r = 1$

5.3 (10 points) **Classical generalized force on a harmonic oscillator.** Consider a harmonic oscillator that is acted on by a generalized force $f(t)$, so that its Hamiltonian becomes

$$\hat{H}(t) = \hat{H}_0 + i\hbar[\hat{a}^\dagger f(t) - \hat{a} f^*(t)] \,,$$

where $\hat{H}_0 = \hbar\omega\hat{a}^\dagger\hat{a}$ is the oscillator's free Hamiltonian (the zero-point energy has been omitted since it has no effect on the dynamics).

The unitary evolution operator for the oscillator satisfies the Schrödinger equation,

$$i\hbar\frac{d\hat{U}(t,0)}{dt} = \hat{H}(t)\hat{U}(t,0) \,,$$

with initial condition $\hat{U}(0,0) = \hat{1}$. The solution for the evolution operator has the form

$$\hat{U}(t,0) = e^{-i\delta(t)}\hat{U}_0(t,0)D\big(\hat{a},\alpha(t)\big) \,.$$

Here $\alpha(t)$ and $\delta(t)$ are functions of time to be determined, $D\big(\hat{a},\alpha(t)\big) = e^{\alpha(t)\hat{a}^\dagger - \alpha^*(t)\hat{a}}$ is the displacement operator, and $\hat{U}_0(t,0) = e^{-(i/\hbar)\hat{H}_0 t} = e^{-i\omega t\hat{a}^\dagger\hat{a}}$ is the free evolution operator.

(a) As a first step in solving for $\hat{U}(t,0)$, *show* that

$$\frac{dD\big(\hat{a},\alpha(t)\big)}{dt} = \big[-\tfrac{1}{2}(\alpha^*\dot{\alpha} - \alpha\dot{\alpha}^*) + (\dot{\alpha}\hat{a}^\dagger - \dot{\alpha}^*\hat{a})\big]D\big(\hat{a},\alpha(t)\big) \,.$$

(Hint: Use the Baker-Campbell-Hausdorff identity to write the displacement operator in normally ordered form, i.e., to put the creation operators to the left of the annihilation operators, and then differentiate.)

(b) Use the result of part (a) to *show* that the Schrödinger equation implies two ordinary differential equations for $\alpha(t)$ and $\delta(t)$, whose solutions are

$$\alpha(t) = \int_0^t dt'\, f(t')e^{i\omega t'}\ ,$$

$$\delta(t) = \tfrac{1}{2}i \int_0^t dt'\, (\alpha^*\dot{\alpha} - \alpha\dot{\alpha}^*)\ .$$

(c) Suppose that the oscillator is initially in the ground state, i.e., $|\psi(0)\rangle = |\varphi_0\rangle$. At time t, *find* the expectation values of $\hat{a}$, $\hat{a}^\dagger$, $\hat{a}^\dagger\hat{a}$, and $\hat{a}^2$ in terms of $\alpha(t)$ and $\delta(t)$. Use these expectation values to find the expectation values and variances of $\hat{x}$ and $\hat{p}$.

5.4 (10 points) Challenge problem (b)