Phys 521 Quantum Mechanics I **Fall 2011**

Homework Assignment #7 **Due Tuesday, November 29**
(50 points) **(at lecture)**

7.5 (10 points) Challenge problem. **Adding spin-1/2 angular momenta.** Consider $N = 2J$ spin-1/2 particles. The Hilbert space for all N particles has dimension 2^N. If we let

$$|j = \tfrac{1}{2}, m = \tfrac{1}{2}\epsilon\rangle \equiv |\epsilon\rangle_{\mathbf{e}_z} \equiv |\epsilon\rangle \ , \quad \epsilon = \pm,$$

be the spin-up and spin-down states for a single particle, then the Hilbert space of all N particles is spanned by an orthonormal basis consisting of the states $|\epsilon_1 \ldots \epsilon_N\rangle$.

The total angular momentum of all N particles is

$$\mathbf{J} = \sum_{l=1}^{N} \mathbf{S}_l = \frac{1}{2}\hbar \sum_{l=1}^{N} \boldsymbol{\sigma}_l \ .$$

Once you get serious about adding angular momenta, perhaps in the second semester of this course, you will learn that the overall Hilbert space can be divided into orthogonal total-angular-momentum subspaces of all integral (N even) or half-integral (N odd) values up to $J = N/2$. Indeed, it is not difficult to determine that the number of subspaces of total angular momentum j is

$$n_{Jj} = \binom{2J-1}{J+j-1} - \binom{2J-1}{J-j-2} = \frac{(2J-1)!}{(J+j-1)!(J-j)!} - \frac{(2J-1)!}{(J-j-2)!(J+j+1)!} \ .$$

This gives the interesting identity

$$\sum_j (2j+1) n_{Jj} = 2^N \ .$$

Fortunately, in this problem, we are only interesting in the single subspace of maximal total angular momentum $j = J = N/2$.

(a) *Show* that

$$|JM\rangle = \sqrt{\frac{n_+!\, n_-!}{N!}} \sum_{\substack{\epsilon_1,\ldots,\epsilon_N \\ M=(n_+-n_-)/2}} |\epsilon_1 \ldots \epsilon_N\rangle \equiv |n_+, n_-\rangle \ ,$$

where n_+ is the number of particles with spin up and n_- is the number with spin down in the sequence $\epsilon_1 \ldots, \epsilon_N$. Since $N = n_+ + n_-$, the sum is restricted to those states that have $(n_+ - n_-)/2 = n_+ - N/2 = N/2 - n_- = M$. (Hint: This is easy. If you're making it hard, you need to think some more.)

The states $|JM\rangle \equiv |n_+, n_-\rangle$ are all completely symmetric under interchange of particles. Hence all the states in this angular-momentum subspace are symmetric under particle interchange, and the subspace itself is called the *symmetric subspace* of the N particles. The main point here is that we can model any angular-momentum subspace J as the symmetric subspace of $N = 2J$ spin-1/2 particles.

(b) Specialize now to $N = 2$ particles. The three states $|J = 1, M = 1\rangle$, $|J = 1, M = 0\rangle$, and $|J = 1, M = -1\rangle$ are called the *triplet*. *Write* out the triplet states in terms of the states $|\epsilon_1, \epsilon_2\rangle$. There is one further state that is orthogonal to all three triplet states. *Find* such a (normalized) state, and show that it can be taken to be the state $|J = 0, M = 0\rangle$; this state is called the *singlet*.