Phys 521 Quantum Mechanics I **Fall 2011**

Homework Assignment #8 **Due Tuesday, December 6**
(40 points) **(at lecture)**

8.1 (10 points) CT $\mathrm{F_{VI}}.8$

8.2 (10 points) CT $\mathrm{F_{VI}}.10$

8.3 (10 points) The Hamiltonian of an isotropic three-dimensional harmonic oscillator is given by

$$H = \frac{\mathbf{P}^2}{2\mu} + \frac{1}{2}\mu\omega^2\mathbf{R}^2 = \hbar\omega\left(a_x^\dagger a_x + a_y^\dagger a_y + a_z^\dagger a_z + \frac{3}{2}\right) = \hbar\omega\left(a_R^\dagger a_R + a_L^\dagger a_L + a_z^\dagger a_z + \frac{3}{2}\right) .$$

Here a_x, a_y, and a_z are annihilation operators for linear oscillators along the three Cartesian axes, and

$$a_R = \frac{1}{\sqrt{2}}\big(a_x - ia_y\big) \quad \text{and} \quad a_L = \frac{1}{\sqrt{2}}\big(a_x + ia_y\big)$$

are annihilation operators for right- and left-circular oscillators. The energy eigenstates generated by a_R, a_L, and a_z are

$$|\chi_{n_R,n_L,n_z}\rangle = \frac{(a_R^\dagger)^{n_R}(a_L^\dagger)^{n_L}(a_z^\dagger)^{n_z}}{\sqrt{n_R!\,n_L!\;n_z!}}|\chi_{0,0,0}\rangle .$$

It is not hard to show that the components of angular momentum, when written out in terms of creation and annihilation operators for the three linear oscillators, are given by

$$L_j = \epsilon_{jkl}X_kP_l = i\hbar\epsilon_{jkl}a_l^\dagger a_k .$$

Written out in full, these relations become

$$\begin{aligned} L_z &= i\hbar(a_y^\dagger a_x - a_x^\dagger a_y) , \\ L_x &= i\hbar(a_z^\dagger a_y - a_y^\dagger a_z) , \\ L_y &= i\hbar(a_x^\dagger a_z - a_z^\dagger a_x) . \end{aligned}$$

The angular-momentum raising and lowering operators thus take the form

$$\begin{aligned} L_+ &= \sqrt{2}\hbar(a_z^\dagger a_L - a_R^\dagger a_z) , \\ L_- &= \sqrt{2}\hbar(a_L^\dagger a_z - a_z^\dagger a_R) . \end{aligned}$$

In this problem, we consider the six-dimensional subspace spanned by the energy eigenstates that have energy $\frac{7}{2}\hbar\omega$.

(a) *Show* that the energy eigenstates $|\chi_{n_R,n_L,n_z}\rangle$ in this six-dimensional subspace are eigenstates of L_z, and *find* the eigenvalues.

(b) In the six-dimensional subspace, *find* eigenstates and corresponding eigenvalues of $\mathbf{L}^2$ and L_z in terms of the states $|\chi_{n_R,n_L,n_z}\rangle$. *Justify* your answers (in particular, you must explain carefully the eigenvalues you assign to $\mathbf{L}^2$).

8.4 (10 points) Challenge problem