Phys 521 Quantum Mechanics I **Fall 2011**

Homework Assignment #8 **Due Tuesday, December 6**
(40 points) **(at lecture)**

8.4 (10 points) Challenge problem. **Schwinger representation.** The *Schwinger representation* is built on a formal identity between angular-momentum operators and bosons with two possible states. Typically the Schwinger representation is developed in terms of photon polarization, where the two states are a photon's two orthogonal polarization states, but it works the same for any two bosonic modes.

A photon is a boson with helicity (spin along the direction of propagation) equal to $\pm\hbar$. Photons with positive helicity (right-handed circular polarization) are created by a creation operator $a^\dagger_{+z}$, and photons with negative helicity (left-handed circular polarization) are created by a creation operator $a^\dagger_{-z}$. The reason for the z in the subscripts comes from thinking in terms of the Poincaré sphere. These creation operators and the corresponding annihilation operators satisfy the canonical bosonic (harmonic-oscillator) commutation relations:

$$[a_{\epsilon z}, a_{\epsilon' z}] = 0, \qquad [a_{\epsilon z}, a^\dagger_{\epsilon' z}] = \delta_{\epsilon\epsilon'} \ .$$

The operator for the total number of photons is

$$N = a^\dagger_{+z}a_{+z} + a^\dagger_{-z}a_{-z} \ .$$

A photon can also be linearly polarized. Photons with orthogonal linear polarizations can be created by creation operators $a^\dagger_{+x}$ and $a^\dagger_{-x}$; the circular-polarization operators are given by

$$a^\dagger_{\pm z} = \frac{1}{\sqrt{2}}(a^\dagger_{+x} \pm ia^\dagger_{-x}) \qquad \Longleftrightarrow \qquad \begin{aligned} a^\dagger_{+x} &= \frac{1}{\sqrt{2}}(a^\dagger_{+z} + a^\dagger_{-z}) \\ a^\dagger_{-x} &= -\frac{i}{\sqrt{2}}(a^\dagger_{+z} - a^\dagger_{-z}) \end{aligned} \ ,$$

i.e., equal linear combinations of the two linear polarizations with a $\pi/2$ phase shift to give circular polarization. To complete the picture, we can introduce operators for the two orthogonal linear polarizations that lie at 45° to the $+x$ and $-x$ polarizations:

$$a^\dagger_{\pm y} = \frac{1}{\sqrt{2}}(\pm a^\dagger_{+x} + a^\dagger_{-x}) \qquad \Longleftrightarrow \qquad \begin{aligned} a^\dagger_{+y} &= \frac{1}{\sqrt{2}}(a^\dagger_{+z}e^{-i\pi/4} + a^\dagger_{-z}e^{i\pi/4}) \\ a^\dagger_{-y} &= -\frac{i}{\sqrt{2}}(a^\dagger_{+z}e^{-i\pi/4} - a^\dagger_{-z}e^{i\pi/4}) \end{aligned} \ .$$

The use of $\pm x$, $\pm y$, and $\pm z$ to label these three polarizations corresponds to using the Poincaré (Bloch) sphere to describe photon polarization. The x, y, and z directions are directions on the Poincaré sphere and are not directions in ordinary three-dimensional space.

We now introduce the operators

$$J_z = \frac{1}{2}\hbar(a^\dagger_{+z}a_{+z} - a^\dagger_{-z}a_{-z}) \;,$$
$$J_x = \frac{1}{2}\hbar(a^\dagger_{+x}a_{+x} - a^\dagger_{-x}a_{-x}) = \frac{1}{2}\hbar(a^\dagger_{+z}a_{-z} + a^\dagger_{-z}a_{+z}) \;,$$
$$J_y = \frac{1}{2}\hbar(a^\dagger_{+y}a_{+y} - a^\dagger_{-y}a_{-y}) = -\frac{i}{2}\hbar(a^\dagger_{+z}a_{-z} - a^\dagger_{-z}a_{+z}) \;.$$

We only need the $\pm z$ operators in this problem, so we drop the z in subscripts below.

(a) *Show* that $J_x^2 + J_y^2 + J_z^2 = \hbar^2(N/2+1)N/2$ and that the three operators J_x, J_y, and J_z satisfy the angular-momentum commutation relations. Since a photon has spin-1, $2J_z$ is, in fact, the total angular momentum along the propagation direction, but aside from this, the operators J_x, J_y, and J_z have no connection to the angular momentum in the photon field. What we exploit here is the purely formal connection of these operators to angular momentum.

The state

$$|n_+, n_-\rangle = \frac{(a^\dagger_+)^{n_+}}{\sqrt{n_+!}}\frac{(a^\dagger_-)^{n_-}}{\sqrt{n_-!}}|0,0\rangle$$

has n_+ photons with positive helicity and n_- photons with negative helicity.

(b) *Show* that

$$|n_+, n_-\rangle = |J = (n_+ + n_-)/2, M = (n_+ - n_-)/2\rangle \;.$$

What you have shown is that the state space of N two-state bosons (photons with two polarizations) is identical to a space of total angular momentum $J = N/2$. As we know, this space of total angular momentum J is the space of maximal angular momentum for $N = 2J$ spin-1/2 particles, i.e., the symmetric subspace of these particles. This should not be surprising because the N bosons have two states and their state must be symmetric under particle exchange.

What we get from the Schwinger representation is the ability to apply techniques from harmonic oscillators (boson modes) to angular-momentum problems and vice versa. There is much to be done with this, but we'll leave that for some future occasion.