Physics 521

Homework #3.

Solution Set

3.1. C-T H_{II}.3

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle$$

$$\langle\psi_0| = \frac{1}{\sqrt{2}}\langle u_1| - \frac{i}{2}\langle u_2| + \frac{1}{2}\langle u_3|$$

$$|\psi_1\rangle = \frac{1}{\sqrt{3}}|u_1\rangle + \frac{i}{\sqrt{3}}|u_3\rangle$$

$$\langle\psi_1| = \frac{1}{\sqrt{3}}\langle u_1| - \frac{i}{\sqrt{3}}\langle u_3|$$

(a) Normalization

$$\langle\psi_0|\psi_0\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{i}{2}\right)\left(-\frac{i}{2}\right) + \left(\frac{1}{2}\right)^2 = 1$$

$$\langle\psi_1|\psi_1\rangle = \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{i}{\sqrt{3}}\right)\left(-\frac{i}{\sqrt{3}}\right) = \frac{2}{3} \longleftarrow \text{not normalized}$$

Normalized vector is $|\bar{\psi}_1\rangle = \dfrac{|\psi_1\rangle}{\sqrt{2/3}} = \dfrac{1}{\sqrt{2}}|u_1\rangle + \dfrac{i}{\sqrt{2}}|u_3\rangle$

(b) Projection operators:

$$\hat{P}_{|\psi_0\rangle} = |\psi_0\rangle\langle\psi_0|$$

$$= \left(\frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle\right)\left(\frac{1}{\sqrt{2}}\langle u_1| - \frac{i}{2}\langle u_2| + \frac{1}{2}\langle u_3|\right)$$

$$= \begin{aligned} &\frac{1}{2}|u_1\rangle\langle u_1| - \frac{i}{2\sqrt{2}}|u_1\rangle\langle u_2| + \frac{1}{2\sqrt{2}}|u_1\rangle\langle u_3| \\ &+\frac{i}{2\sqrt{2}}|u_2\rangle\langle u_1| + \frac{1}{4}|u_2\rangle\langle u_2| + \frac{i}{4}|u_2\rangle\langle u_3| \\ &+\frac{1}{2\sqrt{2}}|u_3\rangle\langle u_1| - \frac{i}{4}|u_3\rangle\langle u_2| + \frac{1}{4}|u_3\rangle\langle u_3| \end{aligned}$$

The matrix representation of $\hat{P}_{|\xi\rangle}$ in the u-basis is

$$\begin{pmatrix} \frac{1}{2} & -\frac{i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & \frac{1}{4} & \frac{i}{4} \\ \frac{1}{2\sqrt{2}} & -\frac{i}{4} & \frac{1}{4} \end{pmatrix}$$

Clearly Hermitian

$$\hat{P}_{|\psi_1\rangle} = |\psi_1\rangle\langle\psi_1|$$

$$= \frac{1}{2}\left(|u_1\rangle + i|u_3\rangle\right)\left(\langle u_1| - i\langle u_3|\right)$$

$$= \frac{1}{2}\Big(|u_1\rangle\langle u_1| - i|u_1\rangle\langle u_3|$$

$$+ i|u_3\rangle\langle u_1| + |u_3\rangle\langle u_3| \Big)$$

The matrix representation of $\hat{P}_{|\psi_1\rangle}$ is

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 1 \end{pmatrix}$$

Clearly Hermitian

3.2. C-T $H_{II}.5$

A projection operator P is a Hermitian operator that satisfies $P^2 = P$ or, equivalently, all of whose eigenvalues are 0 or 1.

Let P_1 be a projector onto subspace $\mathcal{E}_1$ and P_2 be a projector onto subspace $\mathcal{E}_2$. $P_1 P_2$ is a projector if and only if $[P_1, P_2] = 0$.

Proof:

Only if. Assume $P_1 P_2$ is a projector. Then, since $P_1 P_2$ is Hermitian, we have

$$P_1 P_2 = (P_1 P_2)^\dagger = P_2^\dagger P_1^\dagger = P_2 P_1 \implies [P_1, P_2] = 0$$

If. Assume $[P_1, P_2] = 0$, which implies that P_1 and P_2 have a simultaneous eigenbasis $|\varphi_n\rangle$, in which

$$P_1 = \sum_n \lambda_n |\varphi_n\rangle\langle\varphi_n| \,, \quad \lambda_n = 0 \text{ or } 1$$

$$P_2 = \sum_n \mu_n |\varphi_n\rangle\langle\varphi_n| , \quad \mu_n = 0 \text{ or } 1$$

$\mathcal{E}_1$ ($\mathcal{E}_2$) is the subspace spanned by eigenvectors $|\varphi_n\rangle$ such that λ_n (μ_n) is 1.

$$P_1 P_2 = \left(\sum_n \lambda_n |\varphi_n\rangle\langle\varphi_n|\right)\left(\sum_m \mu_m |\varphi_m\rangle\langle\varphi_m|\right)$$

$$= \sum_{n,m} \lambda_n \mu_m |\varphi_n\rangle \underbrace{\langle\varphi_n|\varphi_m\rangle}_{\delta_{nm}} \langle\varphi_m|$$

$$= \sum_n \lambda_n \mu_n |\varphi_n\rangle\langle\varphi_n|$$

Since $\lambda_n \mu_n$ is 0 or 1, $P_1 P_2$ is a projector. It projects onto the subspace spanned by eigenvectors $|\varphi_n\rangle$ such that λ_n <u>and</u> μ_n are both 1. This subspace is the intersection of $\mathcal{E}_1$ and $\mathcal{E}_2$.

3.3. $H_{II}.6$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\Rightarrow \quad \sigma_x^n = \begin{cases} \sigma_x, & n \text{ odd} \\ I, & n \text{ even} \end{cases}$$

$$e^{i\alpha\sigma_x} = \sum_{n=0}^{\infty} \frac{1}{n!}(i\alpha\sigma_x)^n$$

$$= I \underbrace{\sum_{n \text{ even}} \frac{1}{n!}(i\alpha)^n}_{= \frac{1}{2}(e^{i\alpha} + e^{-i\alpha}) = \cos\alpha} + \sigma_x \underbrace{\sum_{n \text{ odd}} \frac{1}{n!}(i\alpha)^n}_{\frac{1}{2}(e^{i\alpha} - e^{-i\alpha}) = i\sin\alpha}$$

$$\therefore \boxed{e^{i\alpha\sigma_x} = I\cos\alpha + i\sigma_x \sin\alpha}$$

3.4. $\hat{A} = \frac{1}{2}\left(|u_1\rangle\langle u_1| + |u_2\rangle\langle u_2| - |u_3\rangle\langle u_3| - |u_4\rangle\langle u_4|\right)$

$\hat{B} = \frac{1}{2}\left(|u_1\rangle\langle u_1| - |u_2\rangle\langle u_2| + |u_3\rangle\langle u_3| - |u_4\rangle\langle u_4|\right)$

$\hat{C} = \hat{A} + \hat{B} = |u_1\rangle\langle u_1| - |u_4\rangle\langle u_4|$

$\hat{D} = |u_1\rangle\langle u_1| + |u_2\rangle\langle u_3| + |u_3\rangle\langle u_2| + |u_4\rangle\langle u_4|$

$|\psi(0)\rangle = \frac{1}{2}\left(|u_1\rangle + |u_2\rangle + |u_3\rangle + |u_4\rangle\right)$

(a) Notice that

$|u_2\rangle\langle u_3| + |u_3\rangle\langle u_2| = |v_2\rangle\langle v_2| - |v_3\rangle\langle v_3|$

$|v_2\rangle = \frac{1}{\sqrt{2}}\left(|u_2\rangle + |u_3\rangle\right)$ $\qquad |u_2\rangle = \frac{1}{\sqrt{2}}\left(|v_2\rangle + |v_3\rangle\right)$

$|v_3\rangle = \frac{1}{\sqrt{2}}\left(|u_2\rangle - |u_3\rangle\right)$ $\qquad |u_3\rangle = \frac{1}{\sqrt{2}}\left(|v_2\rangle - |v_3\rangle\right)$

$\hat{A}$: $|u_1\rangle$ $\quad |u_2\rangle$ $\quad |u_3\rangle$ $\quad |u_4\rangle$

$\frac{1}{2}$ $\quad \frac{1}{2}$ $\quad -\frac{1}{2}$ $\quad -\frac{1}{2}$

$\hat{B}$: $|u_1\rangle$ $\quad |u_2\rangle$ $\quad |u_3\rangle$ $\quad |u_4\rangle$

$\frac{1}{2}$ $\quad -\frac{1}{2}$ $\quad \frac{1}{2}$ $\quad -\frac{1}{2}$

$\hat{C}$: $|u_1\rangle$ $\quad |u_2\rangle$ $\quad |u_3\rangle$ $\quad |u_4\rangle$

or

$|v_2\rangle$ $\quad |v_3\rangle$

1 $\quad 0$ $\quad 0$ $\quad -1$

$\hat{D}$:

	$\lvert u_1\rangle$	$\lvert v_2\rangle$	$\lvert v_3\rangle$	$\lvert u_4\rangle$
	1	1	-1	1

CSCO's: $\lvert u\rangle$-rep: $\hat{A}$ & $\hat{B}$, $\hat{A}$ & $\hat{C}$, $\hat{B}$ & $\hat{C}$ $\quad$ $\lvert v\rangle$-rep: $\hat{C}$ & $\hat{D}$

(b) $\lvert\psi(0)\rangle = \frac{1}{2}\left(\lvert u_1\rangle + \lvert u_2\rangle + \lvert u_3\rangle + \lvert u_4\rangle\right) = \frac{1}{2}\lvert u_1\rangle + \frac{1}{\sqrt{2}}\lvert v_2\rangle + \frac{1}{2}\lvert u_4\rangle$

$\hat{A}$: $\underbrace{\lvert u_1\rangle \quad \lvert u_2\rangle}$ $\quad$ $\underbrace{\lvert u_3\rangle \quad \lvert u_4\rangle}$

results	$\frac{1}{2}$	$-\frac{1}{2}$
probs	$\frac{1}{2}$	$\frac{1}{2}$

$\hat{B}$: $\underbrace{\lvert u_1\rangle \quad \lvert u_3\rangle}$ $\quad$ $\underbrace{\lvert u_2\rangle \quad \lvert u_4\rangle}$

results	$\frac{1}{2}$	$-\frac{1}{2}$
probs	$\frac{1}{2}$	$\frac{1}{2}$

$\hat{C}$: $\lvert u_1\rangle$ $\quad$ $\underbrace{\lvert u_2\rangle \quad \lvert u_3\rangle}$ $\quad$ $\lvert u_4\rangle$

results	1	0	-1
probs	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$\hat{D}$: $\underbrace{\lvert u_1\rangle \quad \lvert v_2\rangle \quad \lvert u_4\rangle}$ $\quad$ $\lvert v_3\rangle$

results	1	-1
probs	1	0

(c) $\hat{H} = \hbar\omega\hat{C} = \hbar\omega\left(|u_1\rangle\langle u_1| - |u_4\rangle\langle u_4|\right)$

$$|\psi(t)\rangle = \frac{1}{2}\left(e^{-i\omega t}|u_1\rangle + |u_2\rangle + |u_3\rangle + e^{i\omega t}|u_4\rangle\right)$$

$$= \frac{1}{2}e^{-i\omega t}|u_1\rangle + \frac{1}{\sqrt{2}}|v_2\rangle + \frac{1}{2}e^{i\omega t}|u_4\rangle$$

$\hat{C}$:	$\|u_1\rangle$	$\underbrace{\|u_2\rangle \quad \|u_3\rangle}$	$\|u_4\rangle$
results	1	0	−1
probs	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$\hat{D}$:	$\underbrace{\|u_1\rangle \quad \|v_2\rangle \quad \|u_4\rangle}$	$\|v_3\rangle$
results	1	−1
probs	1	0

(d) $\hat{H} = \hbar\omega\hat{A} = \frac{1}{2}\hbar\omega\left(|u_1\rangle\langle u_1| + |u_2\rangle\langle u_2| - |u_3\rangle\langle u_3| - |u_4\rangle\langle u_4|\right)$

$$|\psi(t)\rangle = \frac{1}{2}\left(e^{-i\omega t/2}|u_1\rangle + \underbrace{e^{-i\omega t/2}|u_2\rangle + e^{i\omega t/2}|u_3\rangle} + e^{i\omega t/2}|u_4\rangle\right)$$

$$= \frac{1}{\sqrt{2}}e^{-i\omega t/2}\left(|v_2\rangle + |v_3\rangle\right) + \frac{1}{\sqrt{2}}e^{i\omega t/2}\left(|v_2\rangle - |v_3\rangle\right)$$

$$= \sqrt{2}\cos(\omega t/2)|v_2\rangle - i\sqrt{2}\sin(\omega t/2)|v_3\rangle$$

$$= \frac{1}{2}e^{-i\omega t/2}|u_1\rangle + \frac{1}{\sqrt{2}}\cos(\omega t/2)|v_2\rangle - \frac{i}{\sqrt{2}}\sin(\omega t/2)|v_3\rangle$$

$$+ \frac{1}{2}e^{+i\omega t/2}|u_4\rangle$$

$\hat{C}$:	$\lvert u_1\rangle$	$\underbrace{\lvert u_2\rangle \quad \lvert u_3\rangle}$	$\lvert u_4\rangle$
results	1	0	-1
probs	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$\hat{D}$:	$\underbrace{\lvert u_1\rangle \quad \lvert v_2\rangle \quad \lvert u_4\rangle}$	$\lvert v_3\rangle$
results	1	-1
probs	$\frac{1}{2}\left[1+\cos^2(\omega t/2)\right]$	$\frac{1}{2}\sin^2(\omega t/2)$
	$= \frac{3}{4} + \frac{1}{4}\cos\omega t$	$\frac{1}{4} - \frac{1}{4}\cos\omega t$

3.5. C-T H_{II}.10

$$\langle x|XP|\psi\rangle = \int dp\,dx' \underbrace{\langle x|X|p\rangle}_{x\langle x|p\rangle = x\frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}px}} \underbrace{\langle p|P|x'\rangle}_{= p\langle p|x'\rangle = p\frac{1}{\sqrt{2\pi\hbar}}e^{-\frac{i}{\hbar}px'}} \langle x'|\psi\rangle$$

$$= x\int dx' \langle x'|\psi\rangle \underbrace{\int \frac{dp}{2\pi\hbar}\, p\, e^{\frac{i}{\hbar}p(x-x')}}$$

$$= \frac{\hbar}{i}\frac{d}{dx}\int \frac{dp}{2\pi\hbar}\, e^{\frac{i}{\hbar}p(x-x')}$$

$$= \frac{\hbar}{i}\frac{d}{dx}\,\delta(x-x')$$

$$= x\,\frac{\hbar}{i}\frac{d}{dx}\underbrace{\int dx'\,\delta(x-x')\langle x'|\psi\rangle}_{\langle x|\psi\rangle = \psi(x)}$$

$$\boxed{\langle x|XP|\psi\rangle = x\frac{\hbar}{i}\frac{d\psi(x)}{dx}}$$

We can also get this directly from

$$\langle x|XP|\psi\rangle = x\langle x|P|\psi\rangle = x\frac{\hbar}{i}\frac{d\psi(x)}{dx}$$

$$\langle x|PX|\psi\rangle = \int dp\, dx' \, \underbrace{\langle x|P|p\rangle}\underbrace{\langle p|X|x'\rangle}\langle x'|\psi\rangle$$

$$\langle x|P|p\rangle = p\langle x|p\rangle = p\frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{i}{\hbar}px}$$

$$\langle p|X|x'\rangle = x'\langle p|x'\rangle = x'\frac{1}{\sqrt{2\pi\hbar}}\, e^{-\frac{i}{\hbar}px'}$$

The only difference from the previous case is that $x \longrightarrow x'$ in the integral

$$\langle x|PX|\psi\rangle = \int dx'\, x'\langle x'|\psi\rangle \underbrace{\int \frac{dp}{2\pi\hbar}\, p\, e^{\frac{i}{\hbar}p(x-x')}}_{\frac{\hbar}{i}\frac{d}{dx}\delta(x-x')}$$

$$= \frac{\hbar}{i}\frac{d}{dx}\underbrace{\int dx'\, \delta(x-x')\, x'\langle x'|\psi\rangle}_{x\langle x|\psi\rangle = x\psi(x)}$$

$$\boxed{\langle x|PX|\psi\rangle = \frac{\hbar}{i}\frac{d}{dx}\left(x\psi(x)\right) = \frac{\hbar}{i}\left(\psi(x) + x\frac{d\psi(x)}{dx}\right)}$$

We can also get this from

$$\langle x|PX|\psi\rangle = \langle x|XP|\psi\rangle - \langle x|\underbrace{[X,P]}_{i\hbar}|\psi\rangle$$

$$= \frac{\hbar}{i}\left(x\frac{d\psi(x)}{dx} + \psi(x)\right)$$

3.6. The mixed matrix elements $\langle x|\hat{O}|p\rangle \equiv O(x,p)$ determine the operator $\hat{O}$ through the expansion

$$\hat{O} = \int dx\, dp\, |x\rangle O(x,p)\langle p| \; .$$

The mixed matrix elements $O(x,p)$ can be regarded as a function of the c-numbers x and p.

The outer-product operator $|x\rangle\langle p|$ is actually a product of operator δ-functions. We can see this from

$$\langle x'|x\rangle\langle p|p'\rangle = \delta(x'-x)\delta(p'-p) = \langle x'|\delta(\hat{x}-x)\delta(\hat{p}-p)|p'\rangle \; ,$$

which tells us that

$$|x\rangle\langle p| = \delta(\hat{x}-x)\delta(\hat{p}-p) \; .$$

This allows us to write $\hat{O}$ as

$$\hat{O} = \int dx\, dp\, O(x,p)\delta(\hat{x}-x)\delta(\hat{p}-p) = O(\hat{x},\hat{p}) \; . \tag{1}$$

In the final form, I have done the integrals over x and p formally, treating the δ functions in the usual way; the result is to replace x and p in the function $O(x,p)$ by the corresponding operators.

We now have $\hat{O}$ as an explicit operator function of $\hat{x}$ and $\hat{p}$, but you might not like the operator δ-functions (what could they mean?), so we can put things in a friendlier form by doing a Fourier transform in both x and p. Notice that

$$\begin{aligned}\delta(x-\hat{x}) &= \int \frac{du}{\sqrt{2\pi\hbar}} e^{-iu(x-\hat{x})/\hbar} \; , \\ \delta(p-\hat{p}) &= \int \frac{dv}{\sqrt{2\pi\hbar}} e^{iv(p-\hat{p})/\hbar} \; .\end{aligned} \tag{2}$$

Plug these into the expression (1) for $\hat{O}$:

$$\begin{aligned}\hat{O} &= \int dx\, dp\, O(x,p) \int \frac{du}{\sqrt{2\pi\hbar}} e^{-iu(x-\hat{x})/\hbar} \int \frac{dv}{\sqrt{2\pi\hbar}} e^{iv(p-\hat{p})/\hbar} \\ &= \int \frac{du\, dv}{2\pi\hbar} e^{iu\hat{x}/\hbar} e^{-iv\hat{p}/\hbar} \int \frac{dx\, dp}{2\pi\hbar} O(x,p) e^{i(vp-ux)/\hbar} \\ &= \int \frac{du\, dv}{2\pi\hbar} \tilde{O}(x,p) e^{iu\hat{x}/\hbar} e^{-iv\hat{p}/\hbar} \; .\end{aligned} \tag{3}$$

Here

$$\tilde{O}(x,p) = \int \frac{dx\, dp}{2\pi\hbar} O(x,p) e^{i(vp-ux)/\hbar}$$

is the two-dimensional Fourier transform of $O(x,p)$.

Comments:

1. You have to keep the operators in the right order in all these expressions, with position always to the left of momentum. This ordering can be traced back to using the matrix elements $\langle x|\hat{O}|p\rangle$ instead of $\langle p|\hat{O}|x\rangle$. The latter would have worked just as well, but the ordering would then have been to keep momentum always to the left of position. This required operator ordering resolves any ambiguities in how to order $\hat{x}$ and $\hat{p}$ in $O(\hat{x},\hat{p})$.
2. The use of $\hbar$s in Eq. (2) and the relative minus sign in the Fourier transforms in Eq. (2) is conventional, in order to get the standard translation operators for position and momentum, but these things are certainly not required.
3. The final answer (3) is an example of a more general construction. The operators $\hat{x}$ and $\hat{p}$ are the generators of the Lie algebra associated with a Lie group called the Weyl-Heisenberg group. The product of the two unitary translation operators, $e^{iu\hat{x}/\hbar}$ and $e^{-iv\hat{p}/\hbar}$, of which we will see much more later, is the most general member of the Weyl-Heisenberg group. What we prove here is that all operators are linear combinations of the unitary operators in the Weyl-Heisenberg group.

3.7.

(a) $[\hat{A},\hat{B}^n] = n\hat{B}^{n-1}[\hat{A},\hat{B}]$

The property is clearly true for $n=0$ and $n=1$.
Assuming it is true for n, we have

$$[\hat{A},\hat{B}^{n+1}] = \hat{A}\hat{B}^{n+1} - \hat{B}^{n+1}\hat{A}$$

$$\begin{aligned}
\hat{A}\hat{B}^{n+1} &= (\hat{A}\hat{B}^n)\hat{B} \\
&= (\hat{B}^n\hat{A} + [\hat{A},\hat{B}^n])\hat{B} \\
&= \hat{B}^n(\hat{B}\hat{A} + [\hat{A},\hat{B}]) + n\hat{B}^{n-1}[\hat{A},\hat{B}]\hat{B} \\
&= \hat{B}^{n+1}\hat{A} + \hat{B}^n[\hat{A},\hat{B}] + n\hat{B}^{n-1}\Big(\hat{B}[\hat{A},\hat{B}] \\
&\qquad + \underbrace{[[\hat{A},\hat{B}],\hat{B}]}_{=0 \text{ by assumption}}\Big) \\
&= \hat{B}^{n+1}\hat{A} + (n+1)\hat{B}^n[\hat{A},\hat{B}]
\end{aligned}$$

$$\therefore\ [\hat{A},\hat{B}^{n+1}] = (n+1)\hat{B}^n[\hat{A},\hat{B}]$$

(b) $\hat{f}(\lambda) = e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}$

$$\begin{aligned}
\frac{d\hat{f}}{d\lambda} &= e^{\lambda\hat{A}}\hat{A}\hat{B}e^{-\lambda\hat{A}} + e^{\lambda\hat{A}}\hat{B}(-\hat{A})e^{-\lambda\hat{A}} \\
&= e^{\lambda\hat{A}}[\hat{A},\hat{B}]e^{-\lambda\hat{A}} \qquad \equiv [\hat{A},\hat{B}]^{(1)}
\end{aligned}$$

Iterating, $\dfrac{d^n\hat{f}}{d\lambda^n} = e^{\lambda\hat{A}}\underbrace{[\hat{A},[\hat{A},\ldots,[\hat{A},\hat{B}]]\ldots]]}_{n \text{ commutators}}e^{-\lambda\hat{A}}$

$$\hat{f}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n \hat{f}}{d\lambda^n}\bigg|_{\lambda=0} \lambda^n = \sum_{n=0}^{\infty} \frac{1}{n!} {}^{(n)}[\hat{A},\hat{B}]\, \lambda^n$$

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{f}(1) = \sum_{n=0}^{\infty} \frac{1}{n!} {}^{(n)}[\hat{A},\hat{B}] = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2}[\hat{A},[\hat{A},\hat{B}]] + \dots$$

(c) $[\hat{A},\hat{B}] = \gamma \hat{B} \implies {}^{(n)}[\hat{A},\hat{B}] = \gamma^n \hat{B}$

$$\implies e^{\hat{A}} \hat{B} e^{-\hat{A}} = \sum_{n=0}^{\infty} \frac{1}{n!} \gamma^n \hat{B} = e^{\gamma} \hat{B}$$

(d) $\hat{f}(\lambda) \equiv e^{\lambda(\hat{A}+\hat{B})} e^{-\lambda\hat{B}} e^{-\lambda\hat{A}}$, $\hat{f}(0) = \hat{1}$

$$\frac{d\hat{f}}{d\lambda} = (\hat{A}+\hat{B})\hat{f} - e^{\lambda(\hat{A}+\hat{B})}\hat{B}\, e^{-\lambda\hat{B}} e^{-\lambda\hat{A}} - e^{\lambda(\hat{A}+\hat{B})} e^{-\lambda\hat{B}}\hat{A}\, e^{-\lambda\hat{A}}$$

$$\left(\hat{B} + \lambda[\hat{A},\hat{B}]\right) e^{\lambda(\hat{A}+\hat{B})}$$

$$e^{\lambda(\hat{A}+\hat{B})}\left(\hat{A} - \lambda[\hat{B},\hat{A}]\right) e^{-\lambda\hat{B}}$$

$$= \left(\hat{A} + \lambda[\hat{B},\hat{A}] - \lambda[\hat{B},\hat{A}]\right) e^{\lambda(\hat{A}+\hat{B})} e^{-\lambda\hat{B}}$$

$$= \hat{A}\, e^{\lambda(\hat{A}+\hat{B})} e^{-\lambda\hat{B}}$$

$$\frac{d\hat{f}}{d\lambda} = (\hat{A}+\hat{B} - \hat{B} - \lambda[\hat{A},\hat{B}] - \hat{A})\hat{f}$$

$$= -\lambda[\hat{A},\hat{B}]\hat{f}$$

$$\hat{f}(\lambda) = e^{-\frac{1}{2}\lambda^2[\hat{A},\hat{B}]}$$

$\lambda = 1$: $e^{\hat{A}+\hat{B}} e^{-\hat{B}} e^{-\hat{A}} = e^{-\frac{1}{2}[\hat{A},\hat{B}]}$

$$\Longrightarrow e^{\hat{A}+\hat{B}} = e^{-\frac{1}{2}[\hat{A},\hat{B}]} e^{\hat{A}} e^{\hat{B}}$$