Green functions for ordinary linear differential equations with constant coefficients

Consider the nth order differential operator

$$\mathcal{L} = \frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + a_{n-2}\frac{d^{n-2}}{dt^{n-2}} + \cdots + a_1\frac{d}{dt} + a_0\ . \tag{1}$$

We want to find the general solution to the differential equation

$$\mathcal{L}[x(t)] = f(t)\ . \tag{2}$$

where the forcing function $f(t)$ turns on at $t = 0$, i.e., $f(t) = 0$ for $t \leq 0$. The general solution will be the sum

$$x(t) = x_h(t) + x_p(t)\ , \tag{3}$$

where (i) $x_h(t)$ is a homogeneous solution, satisfying $\mathcal{L}[x_h(t)] = 0$, with arbitrary initial values for $x_h(t)$ and its first $n-1$ derivatives at $t = 0$ and (ii) $x_p(t)$ is a particular solution, satisfying $\mathcal{L}[x_h(t)] = f(t)$, with the initial conditions that $x_p(t)$ and its first $n-1$ derivatives vanish at $t = 0$. We will assume that we know how to solve for the general homogeneous solution $x_h(t)$ (and, indeed, this is not hard to do).

There are several good ways to solve the problem, all of which determine the Green function corresponding to the differential operator $\mathcal{L}$. One way is to Fourier transform, turning the differential equation into an algebraic equation, which is easily solved. Ensuring the solution is causal requires moving poles off the real axis in the right way, but even after doing this, the Fourier method is not well suited to setting initial conditions at $t = 0$. A better transform approach is to use the Laplace transform, because it automatically ensures causality and sets initial conditions at $t = 0$. To solve the equation directly in the time domain, without using transform methods, a traditional approach is to regard the forcing function as a sequence of impulses, but if one works directly with δ-function impulses, one has to deal with delicate questions of ensuring causality. To avoid these delicate questions, we'll follow yet a different path, that of writing the forcing function as a sum of steps.

We begin by finding the solution $h(t)$ for a constant force of unit strength; i.e., we are looking for the solution $h(t)$ of

$$\mathcal{L}[h(t)] = 1 \quad \text{for } t > 0, \tag{4}$$

with initial conditions that $h(t)$ and its first $n-1$ derivatives vanish at $t = 0$, i.e.,

$$\left.\frac{d^k h(t)}{dt^k}\right|_{t=0} = 0\ , \quad k = 0, 1, \ldots, n-1. \tag{5}$$

Evaluating the differential equation (4) at $t = 0$ then tells us that $d^n h(t)/dt^n|_{t=0} = 1$. Notice that we have

$$\mathcal{L}[h(t) - 1/a_0] = 0 \quad \text{for } t > 0\ , \tag{6}$$

so $h(t) - 1/a_0 = x_h(t)$ is the homogeneous solution whose first $n-1$ derivatives vanish at $t = 0$, but which has initial value $x_h(0) = -1/a_0$.

If we extend $h(t)$ to negative values of t by defining it to be zero there, i.e., $h(t) = 0$ for $t \leq 0$, we can regard it as the solution for a forcing function that turns on to a constant unit strength at $t = 0$, i.e.,

$$\mathcal{L}[h(t)] = \Theta(t) = \begin{cases} 0\,, & t < 0 \\ 1\,, & t > 0 \end{cases}\,, \tag{7}$$

Differentiating (7) once with respect to t shows that $g(t) = dh/dt$ is the solution for a δ impulse at $t = 0$, i.e.,

$$\mathcal{L}[g(t)] = \delta(t)\;. \tag{8}$$

For $t < 0$, $g(t) = 0$, and for $t > 0$, $g(t)$ is the homogeneous solution with initial conditions

$$\left.\frac{d^k g(t)}{dt^k}\right|_{t=0} = \left.\frac{d^{k+1} h(t)}{dt^{k+1}}\right|_{t=0} = \delta_{k,n-1}\;, \qquad k = 0, 1, \ldots, n-1. \tag{9}$$

The solution $g(t)$ for a δ impulse is called the *Green function.*

Now we're ready to return to an arbitrary forcing function $f(t)$. We write

$$f(t) = \int_0^t dt'\,\frac{df(t')}{dt'} = \int_0^\infty \Theta(t-t')\frac{df(t')}{dt'}\;. \tag{10}$$

In the second form we have written $f(t)$ as an integral of constant forcing functions that turn on at times $t' > 0$. Because of the linearity of the differential equation (2), we can write the particular solution as

$$\begin{aligned} x_p(t) &= \int_0^\infty dt'\,h(t-t')\frac{df(t')}{dt'} \\ &= \int_0^t dt'\,h(t-t')\frac{df(t')}{dt'} \\ &= h(t-t')f(t')\Big|_{t'=0}^{t'=t} - \int_0^t dt'\,\frac{dh(t-t')}{dt'}f(t') \\ &= \int_0^t g(t-t')f(t')\;, \end{aligned} \tag{11}$$

where the last form follows from the vanishing of the boundary terms and that

$$-\frac{dh(t-t')}{dt'} = \frac{dh(t-t')}{dt} = g(t-t')\;. \tag{12}$$

It is clear from the first form of (11) that $x_p(t)$ satisfies (2). To check that $x_p(t)$ has the right initial values, we note that in the last form of (11), when we take derivatives, we have to do two things: differentiate with respect to the upper limit of integration, and differentiate the integrand, which is an explicit function of t. Given the initial conditions (9) for $g(t)$, however, it is easy to see that first $n-1$ derivatives of $x_p(t)$ are given by

$$\frac{d^k x_p(t)}{dt^k} = \int_0^t dt'\, \frac{d^k g(t-t')}{dt^k} f(t')\ , \quad k = 0, \ldots, n-1, \tag{13}$$

and thus that

$$\left.\frac{d^k x(t)}{dt^k}\right|_{t=0} = 0\ , \quad k = 0, \ldots, n-1\ . \tag{14}$$

Summary. To find the particular solution $x_p(t)$, all you have to do is to find the Green function $g(t)$, which is the homogeneous solution with initial conditions (9), and then do the integral

$$x_p(t) = \int_0^t g(t-t') f(t')\ . \tag{15}$$

Now let's do two examples. For the first-order differential equation

$$\frac{dx}{dt} + i\omega x = f(t)\ , \tag{16}$$

we have

$$h(t) = \frac{1}{i\omega}(1 - e^{-i\omega t})\ , \qquad g(t) = e^{-i\omega t}\ , \qquad x_p(t) = \int_0^t dt'\, e^{-i\omega(t-t')} f(t')\ . \tag{17}$$

For the second-order differential equation

$$\frac{d^2 x}{dt} + \omega^2 x = f(t)\ , \tag{18}$$

we have

$$h(t) = \frac{1}{\omega^2}(1 - \cos\omega t)\ , \quad g(t) = \frac{1}{\omega}\sin\omega t\ , \quad x_p(t) = \frac{1}{\omega}\int_0^t dt'\ \sin\omega(t-t') f(t')\ . \tag{19}$$