Phys 522 Quantum Mechanics II **Spring 2010**

Exam #2 **Due Saturday, March 27**
(20% of course grade) **(at noon in TA's mailbox)**

This is an examination for Phys 522. Your score will determine 20% of your grade for the course.

This is a take-home, open-book exam. You may use the textbook, your own notes, your own homework assignments, all class handouts, including solution sets for homework assignments, but do not consult books other than our primary textbook. You may use as much time as you want to complete the exam, as long as it is turned in by the deadline. Your completed exam must be solely your own work; do not consult anyone else in doing the exam.

Do your work on sheets of paper separate from the exam. The solution you hand in for a problem should be neat and legible and in a logical order. It should represent your best effort at solving the problem and should not include false starts and detours that led nowhere.

Staple your solution(s) to this page, and label this page with your name. Then *sign and date the pledge below,* and turn in the exam to the TA's mailbox. You may keep the exam question(s).

I have obeyed the rules in taking this exam. In particular, I have not consulted anyone about the exam, and the completed exam is solely my own work.

______________________________ ________________
Signature Date

Phys 522 Quantum Mechanics II **Spring 2010**

Exam #2 **(100 points)**

Problem 1 (100 points) Consider an isotropic two-dimensional harmonic oscillator with mass m and frequency ω, i.e., a particle with Hamiltonian

$$H_{\rm osc} = \frac{P_x^2 + P_y^2}{2m} + \frac{1}{2}m\omega^2(X^2+Y^2) - \hbar\omega = \hbar\omega(a_x^\dagger a_x + a_y^\dagger a_y) ,$$

where

$$a_x = \sqrt{\frac{m\omega}{2\hbar}}\left(X + i\frac{P_x}{m\omega}\right) \qquad \text{and} \qquad a_y = \sqrt{\frac{m\omega}{2\hbar}}\left(Y + i\frac{P_y}{m\omega}\right) ,$$

are the annihilation operators for the x and y components of the oscillator's motion. The $-\hbar\omega$ subtracts the zero-point energy of the two degrees of freedom so that the vacuum state of the oscillator has zero energy.

It is not hard to work out that the z component of the system's angular momentum is given by

$$L_z = XP_y - YP_x = i\hbar(a_y^\dagger a_x - a_x^\dagger a_y) = \hbar(a_+^\dagger a_+ - a_-^\dagger a_-) ,$$

where

$$a_\pm = \frac{1}{\sqrt{2}}(a_x \mp i a_y)$$

are annihilation operators for oscillator modes that describe right-handed and left-handed rotation about the z axis. The oscillator Hamiltonian can be written in terms of these new operators as

$$H_{\rm osc} = \hbar\omega(a_+^\dagger a_+ + a_-^\dagger a_-) .$$

The energy eigenstates can be written as $|n_x, n_y\rangle$, where n_x and n_y are the number of quanta in the x and y oscillations, or as $|n_+, n_-\rangle$, where n_+ and n_- are the number of quanta undergoing right-handed and left-handed rotations about the z axis. These states satisfy

$$H|n_x, n_y\rangle = \hbar\omega(n_x + n_y)|n_x, n_y\rangle = \hbar\omega N|n_x, n_y\rangle ,$$
$$H|n_+, n_-\rangle = \hbar\omega(n_+ + n_-)|n_+, n_-\rangle = \hbar\omega N|n_+, n_-\rangle .$$

Except for the vacuum state, these states have degeneracies since the energy depends only on the total number of quanta, $N = n_x + n_y = n_+ + n_-$.

The position-representation wave functions of the states $|n_x, n_y\rangle$ are the familiar Hermite Gaussians, and the wave functions of the states $|n_+, n_-\rangle$ are Laguerre Gaussians. If you find yourself needing these position-representation wave functions, you're headed down the wrong path. You may write answers in terms of either set of energy eigenstates.

Now suppose the particle has a charge q. We are going to consider two different perturbations in this problem. The first is an electric field of strength $\mathcal{E}$ along the y axis, for which the potential energy is

$$W_e = -q\mathcal{E}Y ,$$

and the second is a magnetic field of strength $\mathcal{B}$ along the z direction, for which the potential energy (in the magnetic-dipole approximation) is

$$W_m = -M_z \mathcal{B} = -\frac{q\mathcal{B}}{2m} L_z = -\Omega L_z \ ,$$

where $M_z = qL_z/2m$ is the z component of the particle's magnetic-dipole moment and $\Omega = q\mathcal{B}/2m$ is a frequency, called a *Larmor frequency*, that expresses the strength of the coupling to the magnetic field.

In case you're wondering whether this problem has any practical use, I can assure you that the situation it describes is very close to what happens in a trap for charged particles called a *Penning trap*.

(a) (20 points) Suppose that only the electric field is turned on, making the total Hamiltonian $H = H_{\rm osc} + W_e$. Use the fact that the electric field displaces the equilibrium position to *find* the exact energy eigenstates in terms of a position-translation operator $T_{\mathbf{a}} = \exp(-i\mathbf{P}\cdot\mathbf{a}/\hbar)$ and the unperturbed eigenstates and to *find* the corresponding exact energy eigenvalues.

(b) (20 points) Suppose that only the magnetic field is turned on, making the total Hamiltonian $H = H_{\rm osc} + W_m$. *Find* the exact energy eigenstates and the corresponding exact energy eigenvalues.

For the remainder of the problem, suppose that both the electric and magnetic fields are turned on, making the total Hamiltonian

$$H = H_{\rm osc} + W_e + W_m \ .$$

We assume that Ω/ω is irrational; this should allow you to draw a firm conclusion about the degeneracy of the eigenvalues found in part (b).

(c) (30 points) Treating W_e as a perturbation on top of the Hamiltonian $H_0 = H_{\rm osc} + W_m$, *find* the energy eigenvalues to second order in the perturbation W_e.

(d) (30 points) Treating W_m as a perturbation on top of the Hamiltonian $H_0 = H_{\rm osc} + W_e$, *find* the energy eigenvalues to first order in the perturbation W_m.

This problem is actually a scam. Since the total Hamiltonian, including both the electric and magnetic fields, is linear or quadratic in creation and annihilation operators (or in position and momentum variables), the whole problem can be solved exactly. If you really want to be sure your answers to (c) and (d) are right, you can find the exact solution and compare.

Useful facts

$$[X, P] = i\hbar \ , \qquad [a, a^\dagger] = 1$$

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$$

$$a|n\rangle = \sqrt{n}|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \qquad a^\dagger a|n\rangle = n|n\rangle$$

$$T_{\mathbf{a}}^\dagger \mathbf{R} T_{\mathbf{a}} = e^{i\mathbf{P}\cdot\mathbf{a}/\hbar}\mathbf{R}e^{-i\mathbf{P}\cdot\mathbf{a}/\hbar} = \mathbf{R} + \mathbf{a}$$