Phys 522 Quantum Mechanics II **Spring 2010**

Exam #3 **Due Saturday, April 24**
(12.5% of course grade) **(at noon in TA's mailbox)**

This is an examination for Phys 522. Your score will determine 12.5% of your grade for the course.

This is a take-home, open-book exam. You may use the textbook, your own notes, your own homework assignments, all class handouts, including solution sets for homework assignments, but do not consult books other than our primary textbook. You may use as much time as you want to complete the exam, as long as it is turned in by the deadline. Your completed exam must be solely your own work; do not consult anyone else in doing the exam.

Do your work on sheets of paper separate from the exam. The solution you hand in for a problem should be neat and legible and in a logical order. It should represent your best effort at solving the problem and should not include false starts and detours that led nowhere.

Staple your solution(s) to this page, and label this page with your name. Then *sign and date the pledge below,* and turn in the exam to the TA's mailbox. You may keep the exam question(s).

I have obeyed the rules in taking this exam. In particular, I have not consulted anyone about the exam, and the completed exam is solely my own work.

______________________________________ ____________________

Signature Date

Phys 522 Quantum Mechanics II Spring 2010

Exam #3 (100 points)

Problem 1 (100 points) Consider a one-dimensional harmonic oscillator with mass m and frequency ω, i.e., a particle with Hamiltonian

$$H_0 = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 = \hbar\omega(a^\dagger a + \tfrac{1}{2}) \ ,$$

where the annihilation operator is given by

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(X + i\frac{P}{m\omega}\right) \ .$$

The oscillator is initially, at time $t = 0$, in the ground state $|0\rangle$.

At time $t = 0$, a parametric interaction is turned on, which changes the Hamiltonian to

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2[1 + g\cos(2\omega t)]X^2 \ .$$

The parametric interaction modulates the resonant frequency at twice the resonant frequency. The strength of the modulation is characterized by the dimensionless constant g. The parametric interaction can be treated as a perturbation,

$$W(t) = \frac{1}{2}gm\omega^2 X^2\cos(2\omega t) \ ,$$

on top of the unperturbed Hamiltonian H_0.

The state of the system at time t can be written in the energy-eigenstate (number) basis as

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} c_n(t)|n\rangle = e^{-i\omega t/2}\sum_{n=0}^{\infty} b_n(t)e^{-in\omega t}|n\rangle \ .$$

In the following we are interested in the state of the oscillator after the parametric interaction has been on for a time τ. Throughout the problem we are only interested in the secular (resonant) terms. These secular terms do not oscillate at frequency ω, and they dominate when $\omega\tau \gg 1$.

(a) (25 points) *Find* the first-order secular contributions to the number-state amplitudes; i.e., *find* the secular terms in $b_n^{(1)}(\tau)$.

(b) (35 points) *Find* the second-order secular contributions to the number-state amplitudes; i.e., *find* the secular terms in $b_n^{(2)}(\tau)$.

(c) (40 points) Go to an interaction picture relative to H_0. *Find* the interaction-picture Hamiltonian, *make* the rotating-wave approximation (i.e., keep only resonant terms), and *integrate* the result to find the interaction-picture evolution operator $U_I(\tau, 0)$. Use this result to *find* the secular contributions to the amplitudes $b_n(t)$ to the same order as in parts (a) and (b). Your results should agree with those of parts (a) and (b).

$$[X, P] = i\hbar \qquad\qquad [a, a^\dagger] = 1$$

$$a|n\rangle = \sqrt{n}|n-1\rangle \qquad\qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \qquad\qquad a^\dagger a|n\rangle = n|n\rangle$$

$$e^{i\theta a^\dagger a} a e^{-i\theta a^\dagger a} = a e^{-i\theta}$$