Phys 522 Quantum Mechanics II **Spring 2010**

Homework Assignment #1 **Due Tuesday, January 26**
(40 points) **(at lecture)**

1.1 (10 points) CT B_{IX}.1

1.2 (10 points) The eigenstates of $\mathbf{S}\cdot\mathbf{u} = \frac{1}{2}\hbar\boldsymbol{\sigma}\cdot\mathbf{u}$ are given by

$$|+\rangle_{\mathbf{u}} = \cos(\theta/2)|+\rangle_z + e^{i\varphi}\sin(\theta/2)|-\rangle_z \ ,$$
$$|-\rangle_{\mathbf{u}} = \sin(\theta/2)|+\rangle_z - e^{i\varphi/2}\cos(\theta/2)|-\rangle_z \ .$$

Consider a rotation by angle α about the unit vector $-\mathbf{e}_x$. This rotation is described classically by a rotation operator $\mathcal{R}_{-\mathbf{e}_x}(\alpha) \equiv \mathcal{R}$ and quantum mechanically by a rotation operator

$$R_{-\mathbf{e}_x}(\alpha) = \exp\!\left(-\frac{i}{\hbar}\alpha\mathbf{S}\cdot(-\mathbf{e}_x)\right) = e^{i\alpha S_x/\hbar} = e^{i\sigma_x\alpha/2} \equiv R \ .$$

Show that

$$R|\epsilon\rangle_z = e^{i\delta_\epsilon}|\epsilon\rangle_{\mathcal{R}\mathbf{e}_z} \ ,$$

and find the phases δ_ϵ.

1.3 (10 points) **Angular momentum as a vector operator.** In this problem you demonstrate directly that angular momentum $\mathbf{J}$ is a vector operator, i.e., that

$$R^\dagger_{\mathbf{u}}(\alpha)\mathbf{J}R_{\mathbf{u}}(\alpha) = \mathbf{u}(\mathbf{u}\cdot\mathbf{J}) - \mathbf{u}\times(\mathbf{u}\times\mathbf{J})\cos\alpha + \mathbf{u}\times\mathbf{J}\sin\alpha \equiv \mathcal{R}_{\mathbf{u}}(\alpha)\mathbf{J} \ .$$

Here $R_{\mathbf{u}}(\alpha) \equiv e^{-i\alpha\mathbf{J}\cdot\mathbf{u}/\hbar}$ is the unitary rotation operator for a rotation by angle α about axis $\mathbf{u}$, and $\mathcal{R}_{\mathbf{u}}(\alpha)$ is the corresponding 3-dimensional orthogonal matrix.

(a) *Show* that $\mathbf{J}$ is a vector operator by applying the angular-momentum commutator algebra to the general formula

$$e^B A e^{-B} = A + [B,A] + \frac{1}{2!}\big[B,[B,A]\big] + \frac{1}{3!}\Big[B,\big[B,[B,A]\big]\Big] + \cdots \ .$$

Part (a) shows that the vectorial character of angular momentum is a consequence solely of the commutator algebra of the components of angular momentum. Thus, if one can show that spin-1/2 angular momentum $\mathbf{S} = \hbar\boldsymbol{\sigma}/2$ is a vector, the property follows for all angular-momentum operators.

(b) *Show* that $\mathbf{S}$ is a vector operator by using the algebra of Pauli operators, i.e., by using

$$e^{-i\alpha\mathbf{S}\cdot\mathbf{u}/\hbar} = e^{-i\boldsymbol{\sigma}\cdot\mathbf{u}\alpha/2} = \cos(\alpha/2)1 - i\sin(\alpha/2)\boldsymbol{\sigma}\cdot\mathbf{u} \ .$$

1.4 (10 points) Challenge problem.