Phys 522 Quantum Mechanics II **Spring 2010**

Homework Assignment #2 **Due Thursday, February 11**
(60 points) **(at lecture)**

2.6 (10 points) Challenge problem.

Schwinger representation. Consider $N = 2J$ spin-1/2 particles. The total angular momentum is

$$\mathbf{J} = \sum_{l=1}^{N} \mathbf{S}_l = \frac{1}{2}\hbar \sum_{l=1}^{N} \boldsymbol{\sigma}_l \ .$$

This collection of particles has total-angular-momentum subspaces of all integral (N even) or half-integral (N odd) values up to J. If we let n_{Jj} denote the number of subspaces of angular momentum j, the rules for adding angular momentum give the recursion relation

$$n_{J+1/2,j} = n_{J,j-1/2} + n_{J,j+1/2} \ .$$

This recursion relation is sufficiently like the recursion relation satisfied by binomial coefficients, but with a boundary at $j = 0$, that it is not hard to guess (and then to verify) the solution:

$$n_{Jj} = \binom{2J-1}{J+j-1} - \binom{2J-1}{J-j-2} = \frac{(2J-1)!}{(J+j-1)!(J-j)!} - \frac{(2J-1)!}{(J-j-2)!(J+j+1)!} \ .$$

You can ponder how to get this result if you wish, but in this problem we're interested in only one of these subspaces, the single subspace of maximum total angular momentum $j = J = N/2$.

(a) *Show* that

$$|JM\rangle = \sqrt{\frac{n_+!\,n_-!}{N!}} \sum_{\substack{\epsilon_1,\ldots\epsilon_N \\ M=(n_+-n_-)/2}} |\epsilon_1,\ldots\epsilon_N\rangle \equiv |n_+,n_-\rangle \ ,$$

where n_+ is the number of particles with spin up and n_- is the number with spin down. Since $N = n_+ + n_-$, the sum is restricted to those states that have $(n_+ - n_-)/2 = n_+ - N/2 = N/2 - n_- = M$. (Hint: This is easy. If you're making it hard, you need to think some more.)

The states $|JM\rangle \equiv |n_+,n_-\rangle$ are all completely symmetric under interchange of particles. Hence all the states in this angular-momentum subspace are symmetric under particle interchange, and the subspace itself is called the *symmetric subspace* of the N particles. The main point here is that we can model any angular-momentum subspace J as the symmetric subspace of $N = 2J$ spin-1/2 particles.

In a subspace of angular momentum J, a rotation $\mathcal{R}_{\mathbf{u}}(\alpha)$ is represented by the matrix

$$D^{(J)}_{M'M}(\mathcal{R}) = \langle JM'|R_{\mathbf{u}}(\alpha)|JM\rangle = \langle JM'|e^{-i\mathbf{u}\cdot\mathbf{J}\alpha/\hbar}|JM\rangle \ .$$

For spin-1/2, we adopt a special notation for the rotation matrix:

$$D^{(1/2)}_{M'M}(\mathcal{R}) = \langle \tfrac{1}{2}, M' = \epsilon'/2| e^{-i\mathbf{u}\cdot\mathbf{S}\alpha/\hbar} |\tfrac{1}{2}, M = \epsilon/2\rangle = \langle \epsilon' | e^{-i\mathbf{u}\cdot\boldsymbol{\sigma}\alpha/2} |\epsilon\rangle \equiv D_{\epsilon'\epsilon}(\mathcal{R}) \; .$$

(b) *Find* the matrix $D_{\epsilon'\epsilon}$ for an arbitrary rotation $\mathcal{R}$.

(c) *Find* the matrix $D^{(J)}_{M'M}$ for a rotation by π about the y axis.

(d) For an arbitrary rotation, *find* the matrix $D^{(1)}_{M'M}$ in terms of the corresponding spin-1/2 rotation matrix. It should be clear how you could extend this result to find the rotation matrix for arbitrary J in terms of the corresponding matrix for spin-1/2.

Now we proceed to the *Schwinger representation*, which is built on a formal connection between angular-momentum operators and bosons with two possible states. Typically the Schwinger representation is developed in terms of photon polarization, where the two states are a photon's two orthogonal polarization states, but it works the same for any two bosonic modes.

A photon is a boson with helicity (spin along the direction of propagation) equal to $\pm\hbar$. Photons with positive helicity (right-handed circular polarization) are created by a creation operator $a^\dagger_{+z}$, and photons with negative helicity (left-handed circular polarization) are created by a creation operator $a^\dagger_{-z}$. The reason for the z in the subscripts becomes apparent shortly. These creation operators and the corresponding annihilation operators satisfy the canonical bosonic (harmonic-oscillator) commutation relations:

$$[a_{\epsilon z}, a_{\epsilon' z}] = 0, \qquad [a_{\epsilon z}, a^\dagger_{\epsilon' z}] = \delta_{\epsilon\epsilon'} \; .$$

The operator for the total number of photons is

$$N = a^\dagger_{+z} a_{+z} + a^\dagger_{-z} a_{-z} \; .$$

A photon can also be linearly polarized. Photons with orthogonal linear polarizations can be created by creation operators $a^\dagger_{+x}$ and $a^\dagger_{-x}$; the circular-polarization operators are given by

$$a^\dagger_{\pm z} = \frac{1}{\sqrt{2}}(a^\dagger_{+x} \pm i a^\dagger_{-x}) \qquad \Longleftrightarrow \qquad \begin{aligned} a^\dagger_{+x} &= \frac{1}{\sqrt{2}}(a^\dagger_{+z} + a^\dagger_{-z}) \\ a^\dagger_{-x} &= -\frac{i}{\sqrt{2}}(a^\dagger_{+z} - a^\dagger_{-z}) \end{aligned} \; ,$$

i.e., equal linear combinations of the two linear polarizations with a $\pi/2$ phase shift to give circular polarization. To complete the picture, we can introduce operators for the two orthogonal linear polarizations that lie at 45° to the $+x$ and $-x$ polarizations:

$$a^\dagger_{\pm y} = \frac{1}{\sqrt{2}}(\pm a^\dagger_{+x} + a^\dagger_{-x}) \qquad \Longleftrightarrow \qquad \begin{aligned} a^\dagger_{+y} &= \frac{1}{\sqrt{2}}(a^\dagger_{+z} e^{-i\pi/4} + a^\dagger_{-z} e^{i\pi/4}) \\ a^\dagger_{-y} &= -\frac{i}{\sqrt{2}}(a^\dagger_{+z} e^{-i\pi/4} - a^\dagger_{-z} e^{i\pi/4}) \end{aligned} \; .$$

The use of $\pm x$, $\pm y$, and $\pm z$ to label these three polarizations corresponds to using the Poincaré (Bloch) sphere to describe photon polarization. The x, y, and z directions are directions on the Poincaré sphere and are not directions in ordinary three-dimensional space.

We now introduce the operators

$$\begin{aligned} J_z &= \frac{1}{2}\hbar(a^\dagger_{+z}a_{+z} - a^\dagger_{-z}a_{-z}) \ , \\ J_x &= \frac{1}{2}\hbar(a^\dagger_{+x}a_{+x} - a^\dagger_{-x}a_{-x}) = \frac{1}{2}\hbar(a^\dagger_{+z}a_{-z} + a^\dagger_{-z}a_{+z}) \ , \\ J_y &= \frac{1}{2}\hbar(a^\dagger_{+y}a_{+y} - a^\dagger_{-y}a_{-y}) = -\frac{i}{2}\hbar(a^\dagger_{+z}a_{-z} - a^\dagger_{-z}a_{+z}) \ . \end{aligned}$$

We only need the $\pm z$ operators in parts (e) and (f), so we drop the z in subscripts in those parts.

(e) *Show* that $J_x^2 + J_y^2 + J_z^2 = \hbar^2(N/2+1)N/2$ and that the three operators J_x, J_y, and J_z satisfy the angular-momentum commutation relations. Since a photon has spin-1, $2J_z$ is, in fact, the total angular momentum along the propagation direction, but aside from this, the operators J_x, J_y, and J_z have no connection to the angular momentum in the photon field. What we exploit here is the purely formal connection of these operators to angular momentum.

The state

$$|n_+, n_-\rangle = \frac{(a^\dagger_+)^{n_+}}{\sqrt{n_+!}}\frac{(a^\dagger_-)^{n_-}}{\sqrt{n_-!}}|0,0\rangle$$

has n_+ photons with positive helicity and n_- photons with negative helicity.

(f) *Show* that

$$|n_+, n_-\rangle = |J = (n_+ + n_-)/2, M = (n_+ - n_-)/2\rangle \ .$$

What you have shown is that the space of angular momentum J can be realized as the Hilbert space for the polarization of $N = 2J$ photons.

What we get from the Schwinger representation is that we can apply the algebra of creation and annihilation operators to angular-momentum problems.

(g) *Show* that

$$R^\dagger_{\mathbf{u}}(\alpha)a^\dagger_\epsilon R_{\mathbf{u}}(\alpha) = \sum_{\epsilon'} D^*_{\epsilon\epsilon'}a^\dagger_{\epsilon'} \ .$$

(h) A quarter-wave plate is a birefringent piece of material in which two orthogonal linear polarizations, say $+x$ and $-x$, accumulate a $\pi/2$ relative phase shift. What unitary operator describes this transformation? How does a quarter-wave plate interchange linearly polarized and circularly polarized light? A half-wave plate is twice as thick so that the same two linear polarizations accumulate a π relative phase shift. What unitary operator describes this transformation?