Phys 522 Quantum Mechanics II **Spring 2010**

Homework Assignment #5 **Due Tuesday, April 20**
(50 points) **(at lecture)**

5.4 (10 points) Challenge problem. **Adiabatic elimination of an excited-state manifold.** Consider a quantum system with Hamiltonian

$$H = H_0 + W(t) = H_0 + W \cos \omega t \ .$$

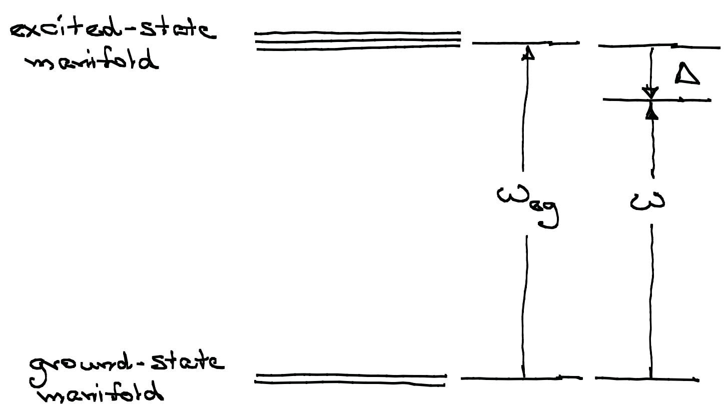
We assume that H_0 has a *ground-state manifold* of nearly degenerate states $|g_j\rangle$ with unperturbed energies $E_{g_j} = \hbar g_j$. We let $E_g = \hbar g$ be a typical ground-state energy and $\hbar\Delta_g$ be the energy difference across the ground-state manifold. We assume that $\omega \gg \Delta_g$.

The perturbation couples the ground-state manifold to a set of states, called the *excited-state manifold*, for which the transition matrix elements of W are nonzero and whose energies are roughly $E_g + \hbar\omega$. The excited states are thus well above the states in the ground-state manifold. We denote the states in the excited-state manifold by $|e_k\rangle$ and their energies by $E_{e_k} = \hbar e_k$. We let $E_e = \hbar e$ be a typical excited-state energy and $\hbar\Delta_e$ be the energy difference across the excited-state manifold. We assume that $\omega \gg \Delta_e$ and that $\omega \simeq \omega_{eg} = e - g$. This leads us to define the detuning $\Delta \equiv \omega - \omega_{eg}$. Finally, we assume that the detuning is much larger than the widths of ground-state and excited-state manifolds, i.e., $|\Delta| \gg \Delta_g, \Delta_e$.

Gathering together our assumptions about frequencies (energies), we have that

$$\Delta_g, \Delta_e \ll |\Delta| \ll \omega_{eg} \simeq \omega = \omega_{eg} + \Delta \ .$$

These assumptions are illustrated in the energy-level diagram below.



Our goal is to develop a description of Hamiltonian dynamics in the ground-state manifold in the presence of the perturbation. Our assumptions allow us to assume that there are no direct (first-order) transitions between ground states, because $\Delta_e \ll \omega$, making ground-state transitions far off resonance, and that there are no direct transitions between ground and excited states, because $|\Delta| \gg \Delta_g, \Delta_e$, making ground-excited state transitions also well off resonance. This means that if the system starts in the ground-state manifold, it stays there; the only effect of the perturbation is to induce virtual transitions back and forth with the excited-state manifold. For short enough times, these virtual transitions can be described wholly within second-order perturbation theory, which describes one cycle of virtual transition to the excited states and back to the ground states. Our strategy is thus to consider the evolution of an arbitrary state in the ground-state manifold for a time τ that is long on the perturbation time scale ω^{-1}, i.e., $\omega\tau \gg 1$, indeed, long on the time scale of the detuning, i.e., $|\Delta|\tau \gg 1$, so that ground-excited transitions become purely virtual, but short on the time scale of the ground-state dynamics, i.e., $\Delta_g\tau \ll 1$. In terms of our hierarchy of frequencies, we can now insert the integration time τ:

$$\Delta_g, \Delta_e \ll 1/\tau \ll |\Delta| \ll \omega_{eg} \simeq \omega = \omega_{eg} + \Delta \ .$$

(a) By considering the evolution of an arbitrary state in the ground-state manifold for a short time from t to $t+\tau$, *show* that evolution in the ground-state manifold is governed by an effective Hamiltonian

$$H_{\text{eff}} = P_g H_0 P_g + \frac{1}{4}\frac{P_g W P_e W P_g}{\hbar\Delta} \ ,$$

where P_g and P_e are the projectors onto the ground-state and excited-state manifolds.

(b) Suppose the system is a charged particle with charge q (e.g., an electron in an atom) and that the perturbation is due to a dipole coupling to an oscillating electric field (could be an electromagnetic wave), i.e., $W(t) = -\mathbf{p}\cdot\mathbf{E}(t) = -\mathbf{p}\cdot\mathbf{E}_0\cos\omega t$ ($W = -\mathbf{p}\cdot\mathbf{E}_0$), where $\mathbf{p} = q\mathbf{R}$ is the particle's electric dipole moment. *Give* a plausible physical interpretation of the effective Hamiltonian in this situation. You will need to think about the time-averaged (over the time τ) energy of the field-induced dipole moment interacting with the applied electric field.