2.7.

Find $\langle JM' | R | JM \rangle = D^{(J)}_{M'M}(R)$ in terms of the spin-$\frac{1}{2}$ rotation matrix $D_{\epsilon'\epsilon}$ for the same rotation R. Getting things in terms of the spin-$\frac{1}{2}$ matrices strongly suggests using the realization of the subspace $|JM\rangle$ as the symmetric subspace of $J = N/2$ spin-$\frac{1}{2}$ particles:

$$|JM\rangle = \sqrt{\frac{n_+!\, n_-!}{N!}} \sum_{\substack{\epsilon_1,\dots,\epsilon_N \\ (n_+ - n_-)/2 = M}} |\epsilon_1,\dots,\epsilon_N\rangle \equiv |n_+, n_-\rangle$$

$$n_+ = N/2 + M = J + M = (\#\text{ of spin-ups in } \epsilon_1,\dots,\epsilon_N)$$
$$n_- = N/2 - M = J - M = (\#\text{ of spin-downs in } \epsilon_1,\dots,\epsilon_N)$$

We can easily apply $R = R_1 \cdots R_N$ to $|\epsilon_1,\dots,\epsilon_N\rangle$, i.e., $R|\epsilon_1,\dots,\epsilon_N\rangle = R_1|\epsilon_1\rangle \otimes \cdots \otimes R_N|\epsilon_N\rangle$, but then there will be some symmetrization required to figure out the desired matrix element. It's easier to use the Schwinger representation, where the creation and annihilation operators take care of the symmetrization automatically.

Schwinger represention: $|JM\rangle = |n_+, n_-\rangle = \dfrac{(a_+^\dagger)^{n_+}(a_-^\dagger)^{n_-}}{\sqrt{n_+!\, n_-!}}|0,0\rangle$

Then we use

$$R a_\epsilon^\dagger R^\dagger = \sum_{\epsilon'} a_{\epsilon'}^\dagger D_{\epsilon'\epsilon} \Leftrightarrow \begin{array}{l} R a_+^\dagger R^\dagger = D_{++} a_+^\dagger + D_{-+} a_-^\dagger \\ R a_-^\dagger R^\dagger = D_{+-} a_+^\dagger + D_{--} a_-^\dagger \end{array}$$

to write

$$R (a_+^\dagger)^{n_+} (a_-^\dagger)^{n_-} R^\dagger = \underbrace{(D_{++} a_+^\dagger + D_{-+} a_-^\dagger)^{n_+}} \underbrace{(D_{+-} a_+^\dagger + D_{--} a_-^\dagger)^{n_-}}.$$

$$= \sum_{k=0}^{n_+} \frac{n_+!}{k!(n_+-k)!} D_{++}^{n_+-k} D_{-+}^{k} (a_+^\dagger)^{n_+-k} (a_-^\dagger)^{k}$$

These are binomial expansions.

$$= \sum_{l=0}^{n_-} \frac{n_-!}{l!(n_--l)!} D_{+-}^{n_--l} D_{--}^{l} (a_+^\dagger)^{n_--l} (a_-^\dagger)^{l}$$

$$R\,(a_+^\dagger)^{n_+}(a_-^\dagger)^{n_-}R^\dagger = \sum_{k=0}^{n_+}\sum_{l=0}^{n_-} \frac{n_+!\,n_-!}{k!(n_+-k)!\,l!(n_--l)!} D_{++}^{n_+-k} D_{-+}^{k} D_{+-}^{n_--l} D_{--}^{l} \times (a_+^\dagger)^{N-k-l}(a_-^\dagger)^{k+l}$$

Now apply this to

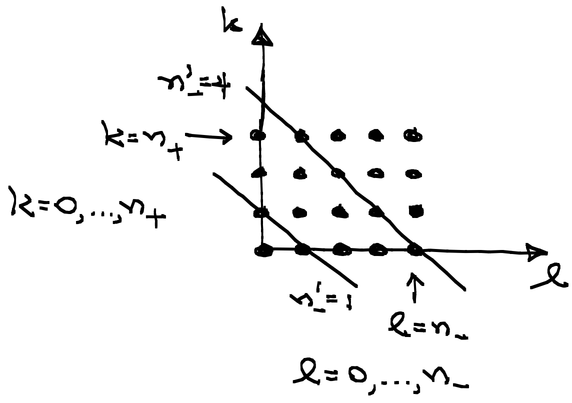
$$R\frac{(a_+^\dagger)^{n_+}(a_-^\dagger)^{n_-}}{\sqrt{n_+!\,n_-!}}|0,0\rangle = R\frac{(a_+^\dagger)^{n_+}(a_-^\dagger)^{n_-}}{\sqrt{n_+!\,n_-!}}R^\dagger \underbrace{R|0,0\rangle}_{|0,0\rangle}$$

$$= \sum_{k=0}^{n_+}\sum_{l=0}^{n_-} \frac{\sqrt{n_+!\,n_-!}}{k!(n_+-k)!\,l!(n_--l)!} D_{++}^{n_+-k} D_{-+}^{k} D_{+-}^{n_--l} D_{--}^{l} \times \underbrace{(a_+^\dagger)^{N-k-l}(a_-^\dagger)^{k+l}|0,0\rangle}_{\sqrt{(N-k-l)!\,(k+l)!}\,\underbrace{|N-k-l,\,k+l\rangle}_{|n'_+,n'_-\rangle = |JM'\rangle}}$$

Now define

$$n'_+ = N-k-l \qquad N = 2J = n'_+ + n'_- \qquad m = (k-l)/2$$

$$n'_- = k+l \qquad M' = (n'_+ - n'_-)/2$$



Now change summing variables from k and l to n'_- and m. Study of the grid to the left shows that the summing ranges are

$$n'_- = 0,\ldots,N$$

$$m = -\underbrace{\min(n'_-/2,\, n_- - n'_-/2)}_{\equiv m_-},\ldots,\underbrace{\min(n'_-/2,\, n_+ - n'_-/2)}_{\equiv m_+}$$

$$R|n_+,n_-\rangle = \sum_{n'_-}|n'_+,n'_-\rangle \sum_{m=-m_-}^{m_+} \frac{\sqrt{n_+!\,n_-!\,n'_+!\,n'_-!}}{k!(n_+-k)!\,l!(n_--l)!} D_{++}^{n_+-k} D_{-+}^{k} D_{+-}^{n_--l} D_{--}^{l}$$

$$k = n'_-/2 + m \qquad n_+ - k = n_+ - n'_-/2 - m$$

$$l = n'_-/2 - m \qquad n_- - l = n_- - n'_-/2 + m$$

$$= \sum_{n'_-} \sqrt{n_+!\, n_-!\, n'_+!\, n'_-!}\, |n'_+, n'_-\rangle$$

$$\times \sum_{m=-m_-}^{m_+} \frac{D_{++}^{n_+ - n'_-/2 - m} D_{-+}^{n'_-/2+m} D_{+-}^{n_- - n'_-/2+m} D_{--}^{n'_-/2-m}}{(n'_-/2+m)!\,(n_+ - n'_-/2-m)!\,(n'_-/2-m)!\,(n_- - n'_-/2+m)!}$$

So

$$\langle JM' | R | JM \rangle = \langle n'_+, n'_- | R | n_+, n_- \rangle$$

$$= \sqrt{n_+!\, n_-!\, n'_+!\, n'_-!}$$

$$\times \sum_{m=-m_-}^{m_+} \frac{D_{++}^{n_+ - n'_-/2 - m} D_{-+}^{n'_-/2+m} D_{+-}^{n_- - n'_-/2+m} D_{--}^{n'_-/2-m}}{(n'_-/2+m)!\,(n_+ - n'_-/2-m)!\,(n'_-/2-m)!\,(n_- - n'_-/2+m)!}$$

$$n_\pm = J \pm M$$

$$n'_\pm = J \pm M'$$

$$m_\pm \equiv \min(n'_-/2,\, n_\pm - n'_-/2)$$