Phys 522

Lectures 9-10

Symmetries and conservation laws

Symmetry operations (transformations)

A <u>symmetry operation</u> is a map $\mathcal{M}: |\psi\rangle \to \mathcal{M}(|\psi\rangle) \equiv |\psi'\rangle$ from normalized vectors to normalized vectors that satisfies

$$|\langle\phi'|\psi'\rangle| = |\langle\phi|\psi\rangle| \quad \longleftarrow \text{ an isometry}$$

for all normalized vectors $|\psi\rangle$ and $|\phi\rangle$.

<u>Wigner's theorem.</u> Given a symmetry operation $\mathcal{M}$, there exists a unitary or anti-unitary operator $\mathcal{U}$ (defined on all of Hilbert space) that agrees with $\mathcal{M}$ up to a phase, i.e.,

$$\mathcal{U}|\psi\rangle = e^{i\alpha(|\psi\rangle)}\mathcal{M}(|\psi\rangle).$$

Antilinear operators

An <u>antilinear operator</u> $\mathcal{K}: |\psi\rangle \to \mathcal{K}|\psi\rangle$ acts on linear combinations according to

$$\mathcal{K}(c_1|\psi_1\rangle + c_2|\psi_2\rangle) = c_1^* \mathcal{K}|\psi_1\rangle + c_2^* \mathcal{K}|\psi_2\rangle$$

Products: A product of linear and antilinear operators is linear (antilinear) if the number of antilinear operators in the product is even (odd).

Left action: $(\langle\phi|\mathcal{K})|\psi\rangle = [\langle\phi|(\mathcal{K}|\psi\rangle)]^*$

$$\Longrightarrow (\langle\phi_1|c_1^* + \langle\phi_2|c_2^*)\mathcal{K} = c_1\langle\phi_1|\mathcal{K} + c_2\langle\phi_2|\mathcal{K}$$

Adjoint: $(\mathcal{K}|\psi\rangle)^\dagger = \langle\psi|\mathcal{K}^\dagger \iff \langle\phi|(\mathcal{K}|\psi\rangle) = \langle\psi|(\mathcal{K}^\dagger|\phi\rangle)$

Adjoints of products follow the standard rule.

Given an orthonormal basis $|e_j\rangle$, we can always define a linear operator A that acts the same as $\mathcal{K}$ on the basis vectors, i.e.,

$$A|e_j\rangle = \mathcal{K}|e_j\rangle$$

On a general vector $|\psi\rangle = \sum_j c_j|e_j\rangle$, we have

$$\mathcal{K}|\psi\rangle = \mathcal{K}\Big(\sum_j c_j|e_j\rangle\Big) = \sum_j c_j^* \mathcal{K}|e_j\rangle = \sum_j c_j^* A|e_j\rangle = A\underbrace{\Big(\sum_j c_j^*|e_j\rangle}_{\mathcal{C}|\psi\rangle} = A\mathcal{C}|\psi\rangle$$

$$\Longrightarrow \mathcal{K} = A\mathcal{C}$$

complex conjugation in this basis

To get all antilinear operators, we only need one, complex conjugation $\mathcal{C}$ in a standard basis. Notice that $\mathcal{C} = \mathcal{C}^{-1} = \mathcal{C}^\dagger$.

An <u>anti-unitary operator</u> is an operator $\mathcal{U}$ that satisfies $\mathcal{U}^\dagger\mathcal{U} = I$. Notice that

$$(\langle\phi|\mathcal{U}^\dagger)(\mathcal{U}|\psi\rangle) = \langle\phi|(\mathcal{U}^\dagger\mathcal{U}|\psi\rangle)^* = \langle\phi|\psi\rangle^* ;$$

i.e., an antiunitary operator conjugates inner products.

We only need one anti-unitary symmetry operation, and it is generally chosen to be time reversal Θ.

Example: particle in three dimensions. Θ is complex conjugation in the position basis

$$\Theta|\vec{r}\rangle = |\vec{r}\rangle \quad \Longrightarrow \quad \Theta^{-1}\vec{R}\,\Theta = \vec{R}$$

$$\Theta|\vec{p}\rangle = \Theta\Big(\int d^3r\, |\vec{r}\rangle\langle\vec{r}|\vec{p}\rangle\Big) = \int d^3r\, |\vec{r}\rangle\underbrace{\langle\vec{r}|\vec{p}\rangle^*}_{\langle\vec{r}|-\vec{p}\rangle} = |-\vec{p}\rangle$$

$$\Longrightarrow \Theta^{-1}\vec{P}\,\Theta = -\vec{P}$$

$$H = \frac{\vec{P}^2}{2m} + V(\vec{R}), \quad \Theta^{-1} H\,\Theta = H$$ ⟵ H is time-reversal invariant

Return to this when we discuss discrete symmetries.

Unitaries preserve commutators
$[U^\dagger A U, U^\dagger B U] = U^\dagger [A,B] U$

Unitary symmetries

Let U be a symmetry operation

States: $U|\psi\rangle = |\psi'\rangle$ is the state that results from applying the (active) symmetry operation to system in state $|\psi\rangle$.

Operators (observables): $\langle\psi'|A|\psi'\rangle = \langle\psi|U^\dagger A U|\psi\rangle$
$U^\dagger A U = A'$ is the operator that results from applying the (active) symmetry operation to system

The expectation value $\langle\psi|A|\psi\rangle$ is the physical quantity that changes under the active transformation to $\langle\psi'|A|\psi'\rangle = \langle\psi|A'|\psi\rangle$. One can attribute this to a change of state ($|\psi\rangle \to |\psi'\rangle$) or to a change of operator ($A \to A'$), but not both.

Passive transformations: unitary basis changes.

A is <u>symmetric</u> <u>(invariant)</u> under the transformation U if

$$\langle\psi|A'|\psi\rangle = \langle\psi'|A|\psi'\rangle = \langle\psi|A|\psi\rangle \quad \text{for all } |\psi\rangle.$$

$$\Longleftrightarrow \quad A' = U^\dagger A U = A \;,\ \text{i.e.,}\ [U,A] = 0$$

Often write $U = e^{ih}$ $\qquad [h,A] = 0$

<u>Conservation law</u>: If the system Hamiltonian H is symmetric under $U = e^{ih}$, then $[U,H] = 0 = [h,H]$.

h is conserved: $i\dfrac{d\langle h^k\rangle}{dt} = \langle [h^k, H]\rangle = 0$ or HP

h and H have simultaneous eigenstates.

If $H|E\rangle = E|E\rangle$, then $HU|E\rangle = UH|E\rangle = EU|E\rangle$, i.e., $U|E\rangle$ is an eigenvector of H with eigenvalue E.

Symmetry groups:

Examples

discrete symmetries (finite groups) ↑

continuous symmetries (Lie groups) ↓

① Parity for a single particle: $\mathcal{G} = \{I, \Pi\}$

$$\Pi|\vec{r}\rangle = |-\vec{r}\rangle \Rightarrow \Pi|\vec{p}\rangle = |-\vec{p}\rangle$$

$$\Rightarrow \Pi\vec{R}\Pi = -\vec{R}, \quad \Pi\vec{P}\Pi = -\vec{P}$$

② Permutation group for N particles, $\mathcal{P}_N$

③ Position translations: $\mathcal{G} = \left\{ T_{\vec{a}} = \exp\left(-\frac{i}{\hbar}\vec{P}\cdot\vec{a}\right) \right\}$

$$T_{\vec{a}}^\dagger \vec{R} T_{\vec{a}} = \vec{R} + \vec{a}, \quad T_{\vec{a}}^\dagger \vec{P} T_{\vec{a}} = \vec{P} \qquad T_{\vec{a}}|\vec{p}\rangle = e^{-i\vec{p}\cdot\vec{a}/\hbar}|\vec{p}\rangle$$

$$\Rightarrow T_{\vec{a}}|\vec{r}\rangle = |\vec{r}+\vec{a}\rangle$$

$$T_{\vec{a}}^\dagger X_j T_{\vec{a}} = e^{i\vec{P}\cdot\vec{a}/\hbar} X_j e^{-i\vec{P}\cdot\vec{a}/\hbar} = X_j + \frac{i}{\hbar} a_k \underbrace{[P_k, X_j]}_{-i\hbar\delta_{jk}} = X_j + a_j$$

$$\langle\vec{r}|T_{\vec{a}}|\psi\rangle = \langle\vec{r}-\vec{a}|\psi\rangle = \psi(\vec{r}-\vec{a})$$

Generators: P_x, P_y, P_z Lie algebra: $[P_j, P_k] = 0$

Many particles?

④ Momentum translations: $\mathcal{G} = \left\{ T_{\vec{q}} = \exp\left(+\frac{i}{\hbar}\vec{R}\cdot\vec{q}\right) \right\}$

$$T_{\vec{q}}^\dagger \vec{P} T_{\vec{q}} = \vec{P} + \vec{q} \qquad T_{\vec{q}}^\dagger \vec{R} T_{\vec{q}} = \vec{R} \qquad T_{\vec{q}}|\vec{r}\rangle = e^{i\vec{r}\cdot\vec{q}/\hbar}|\vec{r}\rangle$$

$$\Rightarrow T_{\vec{q}}|\vec{p}\rangle = |\vec{p}+\vec{q}\rangle$$

$$T_{\vec{q}}^\dagger P_j T_{\vec{q}} = e^{-i\vec{R}\cdot\vec{q}/\hbar} P_j e^{i\vec{R}\cdot\vec{q}/\hbar} = P_j - \frac{i}{\hbar} q_k \underbrace{[X_k, P_j]}_{i\hbar\delta_{jk}} = P_j + q_j$$

$$\langle\vec{p}|T_{\vec{q}}|\psi\rangle = \langle\vec{p}-\vec{q}|\psi\rangle = \psi(\vec{p}-\vec{q})$$

Generators: X, Y, Z Lie algebra: $[X_j, X_k] = 0$

Many particles?

⑤ Phase-space translations (Weyl-Heisenberg group):

$$\mathcal{G} = \left\{ D_{\vec{a}\vec{q}} = \exp\left(\frac{i}{\hbar}(\vec{q}\cdot\vec{R} - \vec{a}\cdot\vec{P})\right) \right\}$$

$$D_{\vec{a}\vec{q}}|\vec{r}\rangle = e^{i(\vec{r}+\vec{a}/2)\cdot\vec{q}/\hbar}|\vec{r}+\vec{a}\rangle$$

$$D_{\vec{a}\vec{q}} = e^{-i\vec{a}\cdot\vec{q}/2\hbar} T_{\vec{q}} T_{\vec{a}} = e^{i\vec{a}\cdot\vec{q}/2\hbar} T_{\vec{a}} T_{\vec{q}} \qquad D_{\vec{a}\vec{q}}|\vec{p}\rangle = e^{-i(\vec{p}+\vec{q}/2)\cdot\vec{a}/\hbar}|\vec{p}+\vec{q}\rangle$$

$$D_{\vec{a}\vec{q}}^\dagger \vec{R} D_{\vec{a}\vec{q}} = \vec{R} + \vec{a}, \quad D_{\vec{a}\vec{q}}^\dagger \vec{P} D_{\vec{a}\vec{q}} = \vec{P} + \vec{q}$$

Generators: X_j, P_k Lie algebra: $[X_j, X_k] = 0$

$[P_j, P_k] = 0$

$[X_j, P_k] = i\hbar\,\delta_{jk}$

6 Rotation group: $\mathcal{G} = \left\{ R_{\vec{u}}(\alpha) = \exp\left(-\frac{i}{\hbar}\alpha\,\vec{u}\cdot\vec{J}\right) \right\}$

$$R^\dagger_{\vec{u}}(\alpha)\,\vec{J}\,R_{\vec{u}}(\alpha) = \mathcal{R}\vec{J} = \sum_{j,k} \vec{e}_j\, O_{jk} J_k$$

Generators J_j Lie algebra: $[J_j, J_k] = i\hbar\,\epsilon_{jkl} J_l$

General compact, semi-simple Lie group $\mathcal{G} = \left\{ U_a = \exp\left(-\frac{i}{\hbar}\sum_j a_j G_j\right) \right\}$

Generators G_j Lie algebra: $[G_j, G_k] = \lambda_{jkl}\, G_l$

structure constants

$$U_a^\dagger\, G_j\, U_a = \sum_k D_{jk}(a)\, G_k$$

group representation

This also applies to position and momentum translations, because they are commutative, and to discrete groups and to discrete/continuous hybrids.

→ Group representations $D_{jk}(a) = \langle e_j | U_a | e_k \rangle$ can be broken down into (unitary) irreducible representations $D^{(\alpha)}_{jk}(a)$ by a basis transformation:

$$\begin{array}{c} \\ \alpha=1 \\ \alpha=1 \\ \alpha=2 \\ \alpha=3 \\ \alpha=3 \end{array}
\begin{array}{c} \begin{array}{ccccc} \alpha=1 & \alpha=1 & \alpha=2 & \alpha=3 & \alpha=3 \end{array} \\
\left(\begin{array}{c|c|c|c|c}
D^{(1)} & 0 & 0 & 0 & 0 \\ \hline
0 & D^{(1)} & 0 & 0 & 0 \\ \hline
0 & 0 & D^{(2)} & 0 & 0 \\ \hline
0 & 0 & 0 & D^{(3)} & 0 \\ \hline
0 & 0 & 0 & 0 & D^{(3)}
\end{array}\right) \end{array}
= \bigoplus_\alpha I_\alpha \otimes D^{(\alpha)}$$

n_α is the # of α irreps

$n_\alpha \times n_\alpha$ unit operator

$d_\alpha \times d_\alpha$ irrep

block diagonal form

Schur's Lemma: $[A, U(a)] = 0$ for all $a \Leftrightarrow A = \bigoplus_\alpha A_\alpha \otimes I_\alpha$

matrix rep
why true?

Conservation laws. H is symmetric under G

$[H, G_j] = 0$ for all j

$\Leftrightarrow$ G_j are conserved

$\Leftrightarrow$ $[H, U(a)] = 0$ for all a

$\Leftrightarrow$ $H = \sum_\alpha H_\alpha \otimes I_\alpha$ $\longleftarrow$ Each H_α can be diagonalized. Group elements $U(a)$ operate in irreps, which are degenerate subspaces of H.

Additional degeneracies of H signal that there are additional conserved quantities.

Degeneracies within an irrep: additional symmetry provides additional label to irreps.

Degeneracies between irreps: additional symmetry mixes irreps and makes them bigger [example: Runge-Lenz vector (tensor) for $1/r$ (r^2) central potential].

Discrete symmetries: Much the same can be said of discrete symmetries, but we usually tack them onto other considerations.

$[\Pi, H] = 0 \Rightarrow$ Energy eigenstates are parity eigenstates or degenerate pairs that can be diagonalized to be parity eigenstates.

Time-reversal invariance:

$[\Theta, H] = 0 \Rightarrow$ Energy eigenstates are real or degenerate, complex conjugate pairs that can be diagonalized to be real.

Appendix to Lecture 9

1. A linear operator A is specified by giving the "sandwiches" $\langle\psi|A|\psi\rangle$ for all normalized vectors $|\psi\rangle$.

Proof: A linear operator A is specified by its matrix elements, $A_{jk} \equiv \langle e_j|A|e_k\rangle$, in an orthonormal basis $|e_j\rangle$, i.e.,

$$A = \sum_{j,k} A_{jk}|e_j\rangle\langle e_k| \ . \tag{1}$$

Clearly the diagonal elements, A_{jj}, are sandwiches. For the off-diagonal element A_{jk}, define the normalized vectors $|f_{jk}\rangle \equiv (|e_j\rangle + |e_k\rangle)/\sqrt{2}$ and $|g_{jk}\rangle \equiv (|e_j\rangle + i|e_k\rangle)/\sqrt{2}$. This gives

$$\langle f_{jk}|A|f_{jk}\rangle = \frac{1}{2}(A_{jj} + A_{kk} + A_{jk} + A_{kj}) \ ,$$
$$\langle g_{jk}|A|g_{jk}\rangle = \frac{1}{2}(A_{jj} + A_{kk} + iA_{jk} - iA_{kj}) \ ,$$

so we have

$$A_{jk} = \langle f_{jk}|A|f_{jk}\rangle - i\langle g_{jk}|A|g_{jk}\rangle - \frac{1-i}{2}\big(\langle e_j|A|e_j\rangle + \langle e_k|A|e_k\rangle\big) \ ;$$

i.e., the off-diagonal elements are also given by sandwiches.

Note: This property does not hold in a real vector space. In a real vector space, an antisymmetric operator is one that satisfies $\langle e_j|A|e_k\rangle = -\langle e_k|A|e_j\rangle$ in some orthonormal basis. This implies that $\langle\phi|A|\psi\rangle = -\langle\psi|A|\phi\rangle$ for all vectors $|\psi\rangle$ and $|\phi\rangle$. An antisymmetric operator thus has all sandwiches equal to zero. This means that in the proof above, we had to use complex numbers in an essential way, and it is not hard to see where this occurred.

2. A vector $|\psi\rangle$ is specified up to an overall phase by giving $|\langle\phi|\psi\rangle|$ for all normalized vectors $|\phi\rangle$.

Proof: A vector is specified by its amplitudes in an orthonormal basis, i.e.,

$$|\psi\rangle = \sum_j c_j|e_j\rangle \ .$$

The absolute values of the amplitudes are given directly by $a_j \equiv |c_j| = |\langle e_j|\psi\rangle|$, so the only question is how to specify the phase in $c_j = a_j e^{i\alpha_j}$ whenever $a_j \neq 0$. If all the a_j's are zero, then $|\psi\rangle = 0$. Otherwise, at least one of the c_j's is nonzero; relabel the basis vectors, if necessary, to make c_1 nonzero. Make $c_1 = a_1$ real by using the overall phase freedom to rephase $|\psi\rangle$.

Now consider the vectors $|f_{j,\pm}\rangle \equiv (|e_1\rangle \pm |e_j\rangle)/\sqrt{2}$ for $j \neq 1$. We have

$$|\langle f_{j,\pm}|\psi\rangle| = \frac{1}{\sqrt{2}}\big|a_1 \pm a_j e^{i\alpha_j}\big| = \sqrt{\frac{a_1^2 + a_j^2 \pm 2a_1a_j\cos\alpha_j}{2}} \ .$$

Whenever $a_j \neq 0$, these two quantities determine α_j, so we're finished.

Note: The method of proof makes clear that this property also holds in a real vector space, where the phases can only be ± 1.

3. A linear operator A is specified up to an overall phase by giving $|\langle\phi|A|\psi\rangle|$ for all normalized vectors $|\phi\rangle$ and $|\psi\rangle$.

Proof: A linear operator is specified by its matrix elements in an orthonormal basis, as in Eq. (1) above. The absolute values of the matrix elements are given directly by $a_{j[} \equiv |A_{jk}| = |\langle e_j|A|e_k\rangle|$, so the only question is how to specify the phase in $A_{jl} = a_{jk}e^{i\alpha_{jk}}$ whenever $a_{jk} \neq 0$. If all the a_{jk}'s are zero, then $A = 0$. Otherwise, at least one of the A_{jk}'s is nonzero; relabel the basis vectors, if necessary, to make at least one element of the first row, say, A_{1K} nonzero. Make $A_{1K} = a_{1K}$ real ($\alpha_{1K} = 0$ by using the overall phase freedom to rephase A.

Now consider the vectors $|f_{j,\pm}\rangle \equiv (|e_1\rangle \pm |e_j\rangle)/\sqrt{2}$ for $j \neq 1$ and $|g_{k,\pm}\rangle \equiv (|e_K\rangle \pm |e_k\rangle)/\sqrt{2}$ for $k \neq K$. We have

$$
\begin{aligned}
|\langle f_{j,\pm}|A|e_k\rangle| &= \frac{1}{\sqrt{2}}\left|a_{1k}e^{i\alpha_{1k}} \pm a_{jk}e^{i\alpha_{jk}}\right| = \frac{1}{\sqrt{2}}\left|a_{1k} \pm a_{jk}e^{i(\alpha_{jk}-\alpha_{1k})}\right| , \\
|\langle e_j|A|g_{k,\pm}\rangle| &= \frac{1}{\sqrt{2}}\left|a_{jK}e^{i\alpha_{jK}} \pm a_{jk}e^{i\alpha_{jk}}\right| = \frac{1}{\sqrt{2}}\left|a_{jK} \pm a_{jk}e^{i(\alpha_{jk}-\alpha_{jK})}\right| .
\end{aligned}
$$

In the upper equation, first choose $k = K$, so that $\alpha_{1K} = 0$; this equation then determines α_{jK} for $j \neq 1$. In the lower equation, first choose $j = 1$, so that $\alpha_{1K} = 0$; this equation then determines α_{1k} for $k \neq K$. This done, either equation determines α_{jk} for $j \neq 1$ and $k \neq K$, so we're finished.

Note: Again the method of proof makes it clear that this property holds in real vector spaces.

4. **Wigner's theorem.** Let $\mathcal{M} : |\psi\rangle \to \mathcal{M}(|\psi\rangle) \equiv |\psi'\rangle$ be a map from normalized vectors to normalized vectors that satisfies

$$|\langle\phi'|\psi'\rangle| = |\langle\phi|\psi\rangle|$$

for all normalized vectors $|\psi\rangle$ and $|\phi\rangle$. There exists a unitary or antiunitary map U (defined on all of Hilbert space) that agrees with $\mathcal{M}$ up to a phase, i.e., $U|\psi\rangle = e^{i\alpha(|\psi\rangle)}\mathcal{M}(|\psi\rangle)$.

Proof: Proof in Messiah-II XV.2, but I would like to make it more convincing and simpler.