1.1 Two-bit classical gates.

(a) Show that the most general two-bit classical gate can be written as

\[
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  x' \\
  y'
\end{pmatrix}
= \begin{pmatrix}
  a \\
  b
\end{pmatrix}
\oplus
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
\oplus
\begin{pmatrix}
  p \\
  q
\end{pmatrix}
xy,
\]

where \(a, b, p, q\), and the four entries in the 2 \(\times\) 2 matrix \(M\) can be either 0 or 1.

(b) Show that the most general reversible two-bit classical gate can be written as

\[
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  x' \\
  y'
\end{pmatrix}
= \begin{pmatrix}
  a \\
  b
\end{pmatrix}
\oplus
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
,
\]

where \(a\) and \(b\) can be either 0 or 1, and where \(M\) can be any of the following matrices:

\[
\begin{pmatrix}
  1 & 0 \\
  0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
  0 & 1 \\
  1 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
  1 & 0 \\
  1 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
  1 & 1 \\
  0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
  1 & 1 \\
  1 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
  0 & 1 \\
  1 & 1
\end{pmatrix}.
\]

(c) The constants \(a\) and \(b\) leave the output unchanged or do a NOT on the output for the two bits. Since we can easily understand the effect of \(a\) and \(b\), we now set them to zero. In this situation, characterize the action of the six reversible \(M\) matrices of part (b).