**Phys 571 Quantum Computation** **Fall 2009**

**Homework Problem 1.3** **Due Thursday, September 3**
**(10 points)** **(at lecture)**

1.3 **Universal classical gates.** The universality of NAND and FANOUT for classical circuits (presuming you get IDENTITY for free) can be expressed in terms of Boolean algebraic formulas. The three nonidentity Boolean functions for a single bit can be written as

$$\begin{aligned} &\text{(i)} \quad 1 \oplus x^2 = 1 \oplus x = \bar{x} \qquad \text{(NOT)}\ , \\ &\text{(ii)} \quad 1 \oplus x(1 \oplus x^2) = 1 \oplus x \oplus x^3 = 1\ , \\ &\text{(iii)} \quad 1 \oplus \big(1 \oplus x(1 \oplus x^2)\big)\big(1 \oplus x(1 \oplus x^2)\big) = 1 \oplus 1 = 0\ . \end{aligned}$$

These are all obviously combinations of NAND and FANOUT: using only NAND requires that the formulas be nested iterations of terms of the form $1 \oplus xy$, and FANOUT implies that any bit value can be duplicated to go into as many such terms as desired.

Because we are going to have quite a bit of nesting in this problem, let's adopt the notation $x \circ y = 1 \oplus xy$ for NAND, thereby rendering the above formulas as

$$\begin{aligned} &\text{(i)} \quad x \circ x = \bar{x} \qquad \text{(NOT)}\ , \\ &\text{(ii)} \quad x \circ (x \circ x) = 1\ , \\ &\text{(iii)} \quad \big(x \circ (x \circ x)\big) \circ \big(x \circ (x \circ x)\big) = 0\ . \end{aligned}$$

You have to keep track of all the parentheses when using this notation, but now the reduction to NANDs is that the formula is reduced to nested circle products.

(a) *Write* the algebraic formulas that reduce the qubit gates AND, OR, and XOR to applications of FANOUT and NAND.

(b) The inductive proof of the universality of NAND and FANOUT proceeds by writing an arbitrary Boolean function $f(x_0, x_1, \ldots, x_N)$ on $N+1$ bits as

$$f(x_0, x_1, \ldots, x_N) = \bar{x}_0 f_0(x_1, \ldots, x_N) \oplus x_0 f_1(x_1, \ldots, x_N)\ ,$$

where one assumes that the $N$-bit functions $f_x(x_1, \ldots, x_N) \equiv f(x, x_1, \ldots, x_N)$ can be evaluated using NAND and FANOUT. *Write* the algebraic formula that reduces the evaluation of $f$ to applications of FANOUT and NAND and evaluations of $f_0$ and $f_1$.

The point of this problem is to convince you of why we use circuit notation.