Phys 571 Quantum Computation **Fall 2009**

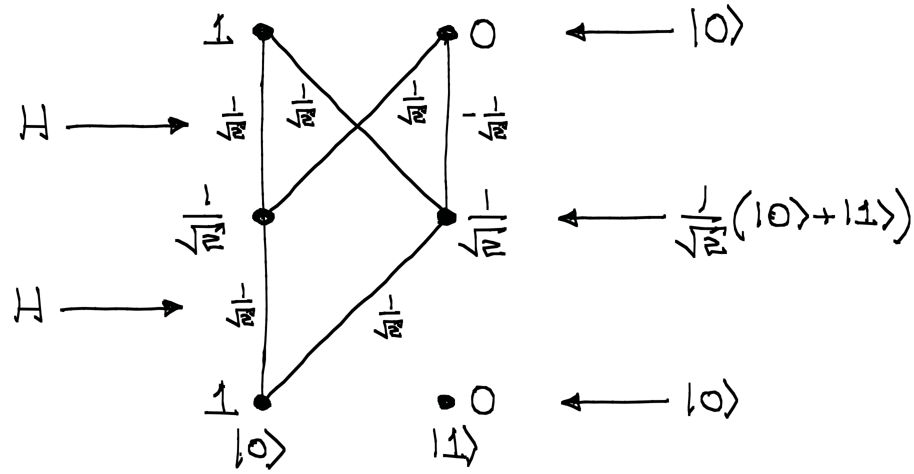
Homework Problem 1.4 **Due Tuesday, September 8**
(10 points) **(at lecture)**

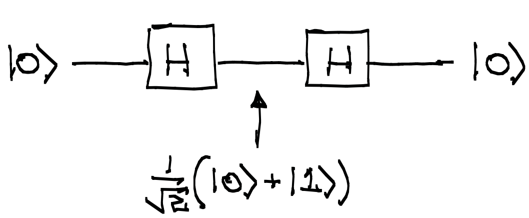
1.4 **Deutsch's algorithm.**

(a) *Draw* two-qubit quantum circuits that calculate each of the four possible 1-bit Boolean functions f, i.e., that take $|x\rangle \otimes |y\rangle$ to $|x\rangle \otimes |y \oplus f(x)\rangle$.

A good way to study interference in a quantum circuit is to diagram the action of the circuit as shown below for the case of two successive Hadamards acting on the initial state $|0\rangle$. In the diagram time runs upward. The nodes on the left are labeled by the amplitude for $|0\rangle$, and the nodes on the right are labeled by the amplitude for $|1\rangle$. A line connecting two nodes is labeled by the amplitude to go from the lower node to the upper node; this is called a *transition amplitude*. The amplitude for a node is obtained by summing over the lines leading to it from the nodes immediately below; the contribution of a line to the sum is the transition amplitude of the line times amplitude of the node from which it originates.

The lines can be hooked up to give paths that run from the initial nodes to the final nodes. The transition amplitude of a path is the product of the transition amplitudes for its lines. The amplitude for a final node is the sum over all paths leading to it, with each path contributing an amplitude equal to the transition amplitude of the path times the amplitude of its initial node. In the case of two Hadamards acting on $|0\rangle$, both final nodes are connected to the initial state $|0\rangle$ by two paths. For final state $|0\rangle$, the two paths interfere constructively, whereas for final state $|1\rangle$, the two paths interfere destructively.





The diagram for a circuit that applies successive unitaries $U_1, \ldots, U_n$ is a graphical representation of the equation

$$\langle x|U_n \cdots U_1|\psi\rangle = \sum_{x_0, x_1, \ldots x_{n-1}} \langle x|U_n|x_{n-1}\rangle \cdots \langle x_1|U_1|x_0\rangle\langle x_0|\psi\rangle \ .$$

In this equation the final amplitude for $|x\rangle$ is written as a sum over all paths leading to $|x\rangle$, with each path contributing an amplitude that is product of the transition amplitudes along the path times the amplitude of the initial state $|x_0\rangle$ of the path. The diagram and the equation are a pedestrian version of Feynman's *path-integral* formulation of quantum mechanics.

(b) *Incorporate* the function-evaluation circuits from part (a) into the circuit for Deutsch's algorithm, and draw an interference diagram for one of the constant functions and one of the balanced functions.

In each of these circuits, the two function values are evaluated twice, all in quantum parallel, by permuting the four computational basis states. All the interference effects are due to the Hadamards.

(c) *Simplify* the four circuits of part (b) by using circuit identities to remove the Hadamards. *Show* how these simplified circuits act on the inputs and thus how the Deutsch algorithm determines the parity of f in one shot.

Part (b) is supposed to leave you with the impression that a lot is going on in a circuit that evaluates a function simultaneously on many inputs using quantum parallelism, but part (c) is supposed to suggest that sometimes this lot is really not much.