Homework Problem 1.5   **Due Tuesday, September 15**
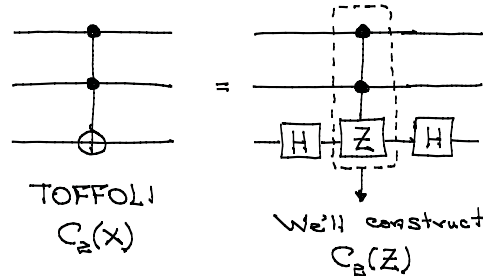**(10 points)**   **(at lecture)**

1.5 **Constructing a circuit for the Toffoli gate.** The Toffoli gate, $C^{(2)}(X)$, can be related to $C^{(2)}(Z)$ by a couple of Hadamard gates.
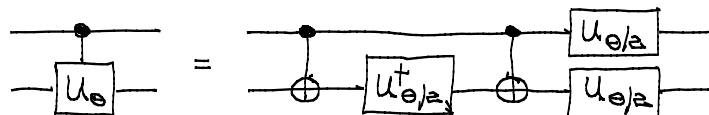

In this problem you will construct a quantum circuit for $C^{(2)}(Z)$, converting it in the end to the Toffoli gate.
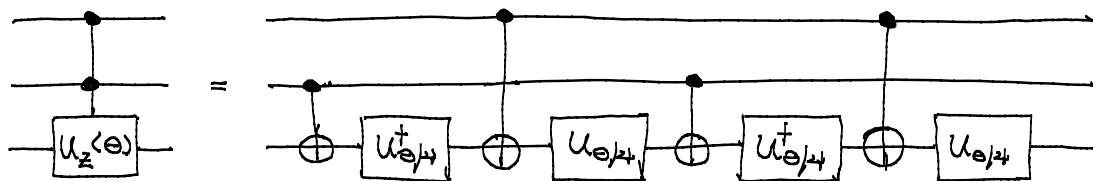
Throughout the problem, let

$$U_\theta \equiv e^{i\theta/2} U_z(\theta) = e^{i\theta/2} e^{-iZ\theta/2} \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} .$$

(a) Using the identity $U_z(\alpha) = X U_z^\dagger(\alpha) X$, *construct* the following circuit for the two-qubit controlled unitary $C^{(1)}(U_\theta)$:

Specialize this circuit to the C-SIGN gate $C^{(1)}(Z)$ and to the controlled-S gate $C^{(1)}(S)$.

(b) Using the same approach as in part (a), *construct* the following circuit for the three-qubit controlled unitary $C^{(2)}\big(U_z(\theta)\big)$:

By generalizing to $C^{(2)}(U_\theta)$ and then specializing to $C^{(2)}(Z)$, *construct* a circuit for the Toffoli gate. *Compare* your result to the circuit in Nielsen and Chuang's Figure 4.9, and *explain* any differences.