3.2 Qudit phase space. For a $D$-dimensional quantum system—a qudit—we have defined “position” eigenstates $|q_j\rangle$ and “momentum” eigenstates $|p_k\rangle$, which are related by discrete Fourier transforms,

\[
|p_k\rangle = \frac{1}{\sqrt{D}} \sum_{j=1}^{D-1} e^{i p_k q_j / \hbar} |q_j\rangle = \frac{1}{\sqrt{D}} \sum_{j=1}^{D-1} e^{2\pi i j k / D} |q_j\rangle,
\]

\[
|q_j\rangle = \frac{1}{\sqrt{D}} \sum_{k=1}^{D-1} e^{-i p_k q_j / \hbar} |p_k\rangle = \frac{1}{\sqrt{D}} \sum_{k=1}^{D-1} e^{-2\pi i j k / D} |p_k\rangle,
\]

where $\hbar = 1/2\pi D$. Now let’s take the next step and define Hermitian “position” and “momentum” operators, $q$ and $p$, in the obvious way:

\[
q |q_j\rangle = q_j |q_j\rangle \quad \text{and} \quad p |p_k\rangle = p_k |p_k\rangle.
\]

We can also define unitary operators that displace the position and momentum eigenstates by one unit, i.e., by $1/D$:

\[
X \equiv e^{-ip/\hbar D} = e^{-2\pi ip} \quad \text{and} \quad Z \equiv e^{iq/\hbar D} = e^{2\pi iq}.
\]

The eigenstates of these operators are the momentum and position and eigenstates:

\[
X |p_k\rangle = e^{-2\pi i k / D} |p_k\rangle \quad \text{and} \quad Z |q_j\rangle = e^{2\pi i j / D} |q_j\rangle.
\]

(a) Show that $X$ and $Z$ displace the position and momentum eigenstates by one unit, i.e., $X |q_j\rangle = |q_{j+1}\rangle$ and $Z |p_k\rangle = |p_{k+1}\rangle$.

(b) Show that $ZX = e^{2\pi i/D} XZ$.

The operator $e^{-i\pi j k / D} Z^k X^j = e^{i\pi j k / D} X^j Z^k$ is a (symmetric) “displacement operator” that displaces by $j$ units in position and $k$ units in momentum. The rest of the problem concentrates on the operator $Y \equiv e^{-i\pi / D} ZX = e^{i\pi / D} XZ$ that displaces by one unit in position and one unit in momentum.

(c) Find the eigenstates $|\psi_l\rangle$ and corresponding eigenvalues $e^{i\phi_l}$ of $Y$. Write the eigenstates in both the position and momentum bases. Along the way, you should establish the Gaussian sum

\[
e^{-i\pi l^2 / D} = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} e^{i\pi j^2 / D} e^{2\pi ijl / D}.
\]

The three bases $\{|q_j\rangle\}$, $\{|p_k\rangle\}$, and $\{|\psi_l\rangle\}$ are called mutually unbiased, because any inner product of a vector from one basis with a vector from another basis has magnitude $1/\sqrt{D}$.

(d) For a qubit ($D = 2$), if we let $|q_a\rangle = |a\rangle$ be the standard basis states, what are the states $|p_a\rangle$ and $|\psi_a\rangle$ and the operators $Z$, $X$, $Y$, and the Fourier transform $F$?