Phys 571 Quantum Computation **Fall 2009**

Homework Problem 3.2 **Due Tuesday, October 27**
(10 points) **(at lecture)**

3.2 **Qudit phase space.** For a D-dimensional quantum system—a *qudit*—we have defined "position" eigenstates $|q_j\rangle$ and "momentum" eigenstates $|p_k\rangle$, which are related by discrete Fourier transforms,

$$|p_k\rangle = \frac{1}{\sqrt{D}}\sum_{j=1}^{D-1} e^{ip_kq_j/\hbar}|q_j\rangle = \frac{1}{\sqrt{D}}\sum_{j=1}^{D-1} e^{2\pi ijk/D}|q_j\rangle\ ,$$

$$|q_j\rangle = \frac{1}{\sqrt{D}}\sum_{k=1}^{D-1} e^{-ip_kq_j/\hbar}|p_k\rangle = \frac{1}{\sqrt{D}}\sum_{k=1}^{D-1} e^{-2\pi ijk/D}|p_k\rangle\ ,$$

where $\hbar = 1/2\pi D$. Now let's take the next step and define Hermitian "position" and "momentum" operators, q and p, in the obvious way:

$$q|q_j\rangle = q_j|q_j\rangle \qquad \text{and} \qquad p|p_k\rangle = p_k|p_k\rangle\ .$$

We can also define unitary operators that displace the position and momentum eigenstates by one unit, i.e., by $1/D$:

$$X \equiv e^{-ip/\hbar D} = e^{-2\pi ip} \qquad \text{and} \qquad Z \equiv e^{iq/\hbar D} = e^{2\pi iq}\ .$$

The eigenstates of these operators are the momentum and position and eigenstates:

$$X|p_k\rangle = e^{-2\pi ik/D}|p_k\rangle \qquad \text{and} \qquad Z|q_j\rangle = e^{2\pi ij/D}|q_j\rangle\ .$$

(a) *Show* that X and Z displace the position and momentum eigenstates by one unit, i.e., $X|q_j\rangle = |q_{j+1}\rangle$ and $Z|p_k\rangle = |p_{k+1}\rangle$.

(b) *Show* that $ZX = e^{2\pi i/D}XZ$.

The operator $e^{-i\pi jk/D}Z^kX^j = e^{i\pi jk/D}X^jZ^k$ is a (symmetric) "displacement operator" that displaces by j units in position and k units in momentum. The rest of the problem concentrates on the operator $Y \equiv e^{-i\pi/D}ZX = e^{i\pi/D}XZ$ that displaces by one unit in position and one unit in momentum.

(c) *Find* the eigenstates $|\psi_l\rangle$ and corresponding eigenvalues $e^{i\phi_l}$ of Y. *Write* the eigenstates in both the position and momentum bases. Along the way, you should establish the Gaussian sum

$$e^{-i\pi l^2/D} = \frac{1}{\sqrt{D}}\sum_{j=0}^{D-1} e^{i\pi j^2/D}e^{2\pi ijl/D}\ .$$

The three bases $\{|q_j\rangle\}$, $\{|p_k\rangle\}$, and $\{|\psi_l\rangle\}$ are called *mutually unbiased*, because any inner product of a vector from one basis with a vector from another basis has magnitude $1/\sqrt{D}$.

(d) For a qubit ($D = 2$), if we let $|q_a\rangle = |a\rangle$ be the standard basis states, what are the states $|p_a\rangle$ and $|\psi_a\rangle$ and the operators Z, X, Y, and the Fourier transform F?