The quantum Fourier transform and the fast Fourier transform.

The quantum Fourier transform (QFT) on \(N\) qubits is a unitary operator defined in the standard (position) basis \(\{|j\rangle, j = 0, \ldots, D - 1\}\) by

\[
F_N|j\rangle = \frac{1}{2^{N/2}} \sum_{k=0}^{D-1} e^{2\pi i jk/D}|k\rangle,
\]

where \(D = 2^N\) is the dimension of the \(N\)-qubit Hilbert space. Since \(j\) and \(k\) are \(N\)-bit binary numbers, i.e.,

\[
j = j_1 \ldots j_N 0 = \sum_{l=1}^{N} j_l 2^{N-l} \quad \text{and} \quad k = k_1 \ldots k_N 0 = \sum_{l=1}^{N} k_l 2^{N-l},
\]

we can write the Fourier-transform coefficient as

\[
e^{2\pi i jk/D} = \prod_{l=1}^{N} e^{2\pi i k_l j/2^l} = \prod_{l=1}^{N} e^{2\pi i k_l 0 \cdot j_{N-l+1} \ldots j_N}.
\]

This allows us to put the QFT in the form

\[
F_N|j\rangle = \frac{1}{2^{N/2}} \bigotimes_{l=1}^{N} \sum_{k_l=0}^{1} |k_l\rangle e^{2\pi i k_l 0 \cdot j_{N-l+1} \ldots j_N} = \frac{1}{2^{N/2}} \bigotimes_{l=1}^{N} \left( |0\rangle + e^{2\pi i 0 \cdot j_{N-l+1} \ldots j_N} |1\rangle \right),
\]

which is suitable for translation into an efficient quantum circuit that does the QFT in \(O(N^2)\) elementary gates.

Applying the QFT to a quantum state \(|\psi\rangle = \sum_j x_j |j\rangle\) changes the amplitudes in the standard basis according to

\[
y_j = \langle j|F_N|\psi\rangle = \sum_{k=0}^{D-1} \langle j|F_N|k\rangle x_k = \frac{1}{2^{N/2}} \sum_{k=0}^{D-1} e^{2\pi i jk/D} x_k.
\]

This transformation is called the discrete Fourier transform (DFT). The QFT doesn’t, of course, compute the DFT, i.e., compute the output amplitudes \(y_k\); it just transforms amplitudes according to the DFT.

To compute the DFT involves taking the matrix product of a \(D \times D\) matrix and a \(D\)-dimensional input vector. A naïve approach to this matrix multiplication requires evaluating \(O(D^2)\) products. Use the identity (I) above to devise an algorithm for computing the DFT using only \(O(ND)\) elementary multiplications. This algorithm is called the fast Fourier transformation (FFT); it is crucial for practical applications of Fourier-transform methods.