Phys 571 Quantum Computation **Fall 2009**

Homework Problem 4.1 **Due Thursday, December 3**
(10 points) **(at lecture)**

4.1 **Alternative derivation of the canonical error operators.** A quantum operation

$$\mathcal{A} = \sum_\alpha A_\alpha \odot A_\alpha^\dagger \ .$$

can be reversed on a code subspace C if and only if the Kraus error operators A_α satisfy

$$\langle e_j | A_\beta^\dagger A_\alpha | e_k \rangle = \mu^2 m_{\alpha\beta} \delta_{jk} \ ,$$

where the vectors $|e_j\rangle$ are any orthonormal basis for C, $\mu^2 \le 1$ ($\mu = 1$ for trace-preserving $\mathcal{A}$), and $m_{\alpha\beta}$ is a trace-one, positive matrix.

The (unnormalized) states that result from errors can be arranged in an array (illustrated here by the case where C is four-dimensional and there are three error operators)

$$\begin{array}{ccc} A_1|e_1\rangle & A_2|e_1\rangle & A_3|e_1\rangle \\ A_1|e_2\rangle & A_2|e_2\rangle & A_3|e_2\rangle \\ A_1|e_3\rangle & A_2|e_3\rangle & A_3|e_3\rangle \\ A_1|e_4\rangle & A_2|e_4\rangle & A_3|e_4\rangle \end{array} \ ,$$

where each row contains the states that arise from all the error operators acting on a particular code-space basis state and each column contains the states that come from a particular error operator acting on all the basis states. The conditions for reversal can be stated in the following way.

> The vectors in a row span a subspace of dimension $\le$ (# of error operators). The subspaces for different rows are orthogonal, and all the rows have the same pairwise inner products, i.e., $\langle e_j | A_\beta^\dagger A_\alpha | e_j \rangle = \mu^2 m_{\alpha\beta}$ is independent of j. As a consequence, the dimensions of all the row subspaces are the same.

These conditions imply that the vectors in a column are orthogonal and thus span a subspace with the dimension of C. All the vectors in a column have the same magnitude $\langle e_j | A_\alpha^\dagger A_\alpha | e_j \rangle^{1/2} = \mu\sqrt{m_{\alpha\alpha}}$. These subspaces are not necessarily orthogonal.

Consider the operator

$$M_1 = \sum_\alpha A_\alpha |e_1\rangle\langle e_1| A_\alpha^\dagger$$

for the first row. *Show* how the eigenvectors $|\tilde{e}_\beta\rangle$ and eigenvalues λ_β of M can be used to define canonical Kraus error operators $\tilde{A}_\beta$ for $\mathcal{A}$, which satisfy

$$\langle e_j | \tilde{A}_\beta^\dagger \tilde{A}_\alpha | e_k \rangle = \mu^2 d_\alpha \delta_{\alpha\beta} \delta_{jk} \ ,$$

where $d_\alpha \ge 0$ and $\sum_\alpha d_\alpha = 1$.