4.1 Alternative derivation of the canonical error operators. A quantum operation
\[ A = \sum_\alpha A_\alpha \otimes A_\alpha^\dagger. \]
can be reversed on a code subspace \( C \) if and only if the Kraus error operators \( A_\alpha \) satisfy
\[ \langle e_j | A_\beta^\dagger A_\alpha | e_k \rangle = \mu^2 m_{\alpha\beta} \delta_{jk}, \]
where the vectors \( |e_j\rangle \) are any orthonormal basis for \( C \), \( \mu^2 \leq 1 \) (\( \mu = 1 \) for trace-preserving \( A \)), and \( m_{\alpha\beta} \) is a trace-one, positive matrix.

The (unnormalized) states that result from errors can be arranged in an array (illustrated here by the case where \( C \) is four-dimensional and there are three error operators)
\[
\begin{array}{ccc}
A_1|e_1\rangle & A_2|e_1\rangle & A_3|e_1\rangle \\
A_1|e_2\rangle & A_2|e_2\rangle & A_3|e_2\rangle \\
A_1|e_3\rangle & A_2|e_3\rangle & A_3|e_3\rangle \\
A_1|e_4\rangle & A_2|e_4\rangle & A_3|e_4\rangle \\
\end{array}
\]
where each row contains the states that arise from all the error operators acting on a particular code-space basis state and each column contains the states that come from a particular error operator acting on all the basis states. The conditions for reversal can be stated in the following way.

The vectors in a row span a subspace of dimension \( \leq \) (\# of error operators).
The subspaces for different rows are orthogonal, and all the rows have the same pairwise inner products, i.e., \( \langle e_j | A_\beta^\dagger A_\alpha | e_j \rangle = \mu^2 m_{\alpha\beta} \) is independent of \( j \). As a consequence, the dimensions of all the row subspaces are the same.

These conditions imply that the vectors in a column are orthogonal and thus span a subspace with the dimension of \( C \). All the vectors in a column have the same magnitude \( \langle e_j | A_\alpha^\dagger A_\alpha | e_j \rangle^{1/2} = \mu \sqrt{m_{\alpha\alpha}} \). These subspaces are not necessarily orthogonal.

Consider the operator
\[ M_1 = \sum_\alpha A_\alpha |e_1\rangle \langle e_1 | A_\alpha^\dagger \]
for the first row. Show how the eigenvectors \( |\tilde{e}_\beta\rangle \) and eigenvalues \( \lambda_\beta \) of \( M \) can be used to define canonical Kraus error operators \( \tilde{A}_\beta \) for \( A \), which satisfy
\[ \langle e_j | \tilde{A}_\beta^\dagger \tilde{A}_\alpha | e_k \rangle = \mu^2 d_\alpha \delta_{\alpha\beta} \delta_{jk}, \]
where \( d_\alpha \geq 0 \) and \( \sum_\alpha d_\alpha = 1 \).