Solution 1.3

Recall that $\mathrm{NOT}[x] = x \circ x$.

(a)

$$\begin{aligned}
\mathrm{AND}[x,y] &= xy = \mathrm{NOT}\big[\mathrm{NAND}[x,y]\big] = (x \circ y) \circ (x \circ y) \ , \\
\mathrm{OR}[x,y] &= x \oplus y \oplus xy = 1 \oplus \bar{x}\bar{y} = \mathrm{NAND}\big[\mathrm{NOT}[x], \mathrm{NOT}[y]\big] = (x \circ x) \circ (y \circ y) \ , \\
\mathrm{XOR}[x,y] &= x \oplus y = (1 \oplus \bar{x}\bar{y})(1 \oplus xy) \\
&= \mathrm{AND}\big[\mathrm{OR}[x,y], \mathrm{NAND}[x,y]\big] \\
&= \Big(\big((x \circ x) \circ (y \circ y)\big) \circ \big(x \circ y\big)\Big) \circ \Big(\big((x \circ x) \circ (y \circ y)\big) \circ \big(x \circ y\big)\Big)
\end{aligned}$$

(b)

$$f(x_0, x_1, \ldots, x_N) = \bar{x}_0 f_0(x_1, \ldots, x_N) \oplus x_0 f_1(x_1, \ldots, x_N) = \mathrm{XOR}\big[\mathrm{AND}[\bar{x}_0, f_0], \mathrm{AND}[x_0, f_1]\big] \ ,$$

so we write

$$\begin{aligned}
&f(x, x_1, \ldots, x_N) \\
&= \Bigg(\Bigg(\Big(\mathrm{AND}[\bar{x}, f_0] \circ \mathrm{AND}[\bar{x}, f_0]\Big) \circ \Big(\mathrm{AND}[x, f_1] \circ \mathrm{AND}[x, f_1]\Big)\Bigg) \circ \Big(\mathrm{AND}[\bar{x}, f_0] \circ \mathrm{AND}[x, f_1]\Big)\Bigg) \\
&\quad \circ \Bigg(\Bigg(\Big(\mathrm{AND}[\bar{x}, f_0] \circ \mathrm{AND}[\bar{x}, f_0]\Big) \circ \Big(\mathrm{AND}[x, f_1] \circ \mathrm{AND}[x, f_1]\Big)\Bigg) \circ \Big(\mathrm{AND}[\bar{x}, f_0] \circ \mathrm{AND}[x, f_1]\Big)\Bigg) \\
&= \Bigg(\Bigg(\Bigg(\Big(\big((x \circ x) \circ f_0\big) \circ \big((x \circ x) \circ f_0\big)\Big) \circ \Big(\big((x \circ x) \circ f_0\big) \circ \big((x \circ x) \circ f_0\big)\Big)\Bigg) \\
&\qquad \circ \Bigg(\Big(\big(x \circ f_1\big) \circ \big(x \circ f_1\big)\Big) \circ \Big(\big(x \circ f_1\big) \circ \big(x \circ f_1\big)\Big)\Bigg)\Bigg) \\
&\quad \circ \Bigg(\Big(\big((x \circ x) \circ f_0\big) \circ \big((x \circ x) \circ f_0\big)\Big) \circ \Big(\big(x \circ f_1\big) \circ \big(x \circ f_1\big)\Big)\Bigg)\Bigg) \\
&\quad \circ \Bigg(\Bigg(\Bigg(\Big(\big((x \circ x) \circ f_0\big) \circ \big((x \circ x) \circ f_0\big)\Big) \circ \Big(\big((x \circ x) \circ f_0\big) \circ \big((x \circ x) \circ f_0\big)\Big)\Bigg) \\
&\qquad \circ \Bigg(\Big(\big(x \circ f_1\big) \circ \big(x \circ f_1\big)\Big) \circ \Big(\big(x \circ f_1\big) \circ \big(x \circ f_1\big)\Big)\Bigg)\Bigg) \\
&\quad \circ \Bigg(\Big(\big((x \circ x) \circ f_0\big) \circ \big((x \circ x) \circ f_0\big)\Big) \circ \Big(\big(x \circ f_1\big) \circ \big(x \circ f_1\big)\Big)\Bigg)\Bigg)
\end{aligned}$$

This formula is nearly unreadable and certainly pointless. It does, however, show three things.

1. The formula has obvious structure, which is captured in a circuit diagram far more effectively than in an algebraic formula.
2. The use of gates built from the fundamental gates—in this case, the use of NOT, AND, and OR, instead of reducing to the fundamental gate NAND—is essential for depicting and understanding even fairly simple circuits.
3. After a sufficiently long time, I can make TEX produce a formula as complicated as this.

We can simplify things pretty radically by using an approach pointed out to me by Mark Olah.

Let's first simplify XOR. Start with de Morgan's rule:

$$1 \oplus x \oplus y \oplus xy = \mathrm{NOR}[x, y] = \mathrm{AND}\big[\mathrm{NOT}[x], \mathrm{NOT}[y]\big] = \bar{x}\bar{y}\ .$$

Then notice that $\mathrm{XOR}[x, y] = \mathrm{OR}\big[\mathrm{AND}[\bar{x}, y], \mathrm{AND}[x, \bar{y}]\big]$, which when combined with de Morgan's rule gives

$$x \oplus y = \mathrm{XOR}[x, y] = \mathrm{NAND}\big[\mathrm{NAND}[\bar{x}, y], \mathrm{NAND}[x, \bar{y}]\big] = \big((x \circ x) \circ y\big) \circ \big(x \circ (y \circ y)\big)\ .$$

This is a considerable simplification of the previous reduction of XOR to NANDs.

Now we're going to do the same thing, but in the slightly more general context of (b). Notice that for any x, y, and z, it is true that

$$\bar{x}y \oplus xz = \mathrm{XOR}\big[\mathrm{AND}[\bar{x}, y], \mathrm{AND}[x, z]\big] = \mathrm{OR}\big[\mathrm{AND}[\bar{x}, y], \mathrm{AND}[x, z]\big]\ ,$$

because the arguments of the XOR and OR cannot simultaneously be true. Thus we have from de Morgan's rule,

$$\begin{aligned} f(x, x_1, \ldots, x_N) &= \bar{x} f_0(x_1, \ldots, x_N) \oplus x f_1(x_1, \ldots, x_N) \\ &= \mathrm{XOR}\big[\mathrm{AND}[\bar{x}, f_0], \mathrm{AND}[x, f_1]\big] \\ &= \mathrm{OR}\big[\mathrm{AND}[\bar{x}, f_0], \mathrm{AND}[x, f_1]\big] \\ &= \mathrm{NAND}\big[\mathrm{NAND}[\bar{x}, f_0], \mathrm{NAND}[x, f_1]\big] \\ &= \big((x \circ x) \circ f_0\big) \circ \big(x \circ f_1\big)\ . \end{aligned}$$

The lesson here might be that the circle notation is a very bad way to recognize ways to reduce the complexity of our formulas, as it just proliferates complicated formulas.