Solution 2.2

(a) Start with

$$|\beta_{ab}\rangle\langle\beta_{ab}| = \frac{1}{2}\sum_{c,d}(-1)^{a(c+d)}|c, c\oplus b\rangle\langle d, d\oplus b| \ .$$

We use

$$\begin{aligned} I &= |0\rangle\langle 0| + |1\rangle\langle 1| \\ Z &= |0\rangle\langle 0| - |1\rangle\langle 1| \\ X &= |0\rangle\langle 1| + |1\rangle\langle 0| \\ iY &= |0\rangle\langle 1| - |1\rangle\langle 0| \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} |0\rangle\langle 0| &= \tfrac{1}{2}(I+Z) \\ |1\rangle\langle 1| &= \tfrac{1}{2}(I-Z) \\ |0\rangle\langle 1| &= \tfrac{1}{2}(X+iY) \\ |1\rangle\langle 0| &= \tfrac{1}{2}(X-iY) \end{aligned}$$

to write

$$|c\rangle\langle d| = \frac{1}{2}\Big(\delta_{cd}\big(I + (-1)^c Z\big) + \delta_{c,1\oplus d}\big(X + i(-1)^c Y\big)\Big) \ .$$

Now it's just a matter of cranking:

$$\begin{aligned} |\beta_{ab}\rangle\langle\beta_{ab}| &= \frac{1}{2}\sum_{c,d}(-1)^{a(c+d)}\Big(\delta_{cd}\big(I + (-1)^c Z\big) + \delta_{c,1\oplus d}\big(X + i(-1)^c Y\big)\Big) \\ &\qquad\qquad \otimes \Big(\delta_{cd}\big(I + (-1)^{c+b} Z\big) + \delta_{c,1\oplus d}\big(X + i(-1)^{c+b} Y\big)\Big) \\ &= \frac{1}{2}\sum_{c}\Big(\big(I + (-1)^c Z\big)\big(I + (-1)^{c+b} Z\big) \\ &\qquad\qquad + (-1)^a\big(X + i(-1)^c Y\big)\big(X + i(-1)^{c+b} Y\big)\Big) \\ &= I\otimes I + (-1)^b Z\otimes Z + (-1)^a X\otimes X - (-1)^{a+b} Y\otimes Y \ . \end{aligned}$$

You can, of course, work out each projector separately, but I like this way of doing it, partially because it means less TEXing for me.

(b) Basically, what we're doing here is inverting a matrix all of whose entries are ± 1. This could be tedious, so I prefer to introduce a binary representation for the Pauli products (this binary representation will later become a mainstay of our discussion of the stabilizer formalism for Pauli products):

$$\Sigma_{00} \equiv I\otimes I \ , \quad \Sigma_{01} \equiv Z\otimes Z \ , \quad \Sigma_{10} \equiv X\otimes X \ , \quad \Sigma_{11} \equiv -Y\otimes Y \ .$$

In terms of this binary representation, the expression for the Bell-state projectors becomes

$$|\beta_{ab}\rangle\langle\beta_{ab}| = \sum_{c,d}(-1)^{ac+bd}\Sigma_{cd} \ .$$

But now we can use the trivial result that inverts the Hadamard transformation,

$$\frac{1}{2}\sum_{b}(-1)^{ab} = \delta_{a0} \ ,$$

to obtain

$$\Sigma_{cd} = \frac{1}{4}\sum_{a,b}(-1)^{ac+bd}|\beta_{ab}\rangle\langle\beta_{ab}|\;,$$

which gives

$$\begin{aligned}
I\otimes I &= \frac{1}{4}\sum_{a,b}|\beta_{ab}\rangle\langle\beta_{ab}| \quad \text{(this has to be true)}\;,\\
Z\otimes Z &= \frac{1}{4}\sum_{a,b}(-1)^{b}|\beta_{ab}\rangle\langle\beta_{ab}|\;,\\
X\otimes X &= \frac{1}{4}\sum_{a,b}(-1)^{a}|\beta_{ab}\rangle\langle\beta_{ab}|\;,\\
Y\otimes Y &= -\frac{1}{4}\sum_{a,b}(-1)^{a+b}|\beta_{ab}\rangle\langle\beta_{ab}|\;.
\end{aligned}$$

These are the expressions given in the assignment.

(c) It is clear from the expressions in (b) that $Z\otimes Z$ and $X\otimes X$ (and $Y\otimes Y$) are diagonal in the Bell basis and that their eigenvalues are doubly degenerate. The eigenvalue of $Z\otimes Z$ gives the parity bit b of the Bell state, and the eigenvalue of $X\otimes X$ gives the phase bit a. A simultaneous measurement of $Z\otimes Z$ and $X\otimes X$ determines both the parity and phase of the Bell state and thus specifies a Bell state.