Solution 2.5

(a)

\[ |2\psi\rangle \xrightarrow{\text{XOX}} Z \xrightarrow{b} X^b Z^a |2\psi\rangle \]

Forget about normalization.

\[ |2\psi\rangle \otimes |0\rangle = \sum_c C_c |c, 0\rangle = \sum_{c, d, e} C_c (-1)^{cd} |d, e\rangle \]

Measurement of XOX projects out case \(e = d\)

for \(e \neq 0\) and \(e = d \otimes 1\) for \(e = 1\), so \(e = d \otimes a\).

\[ \sum_{c, d, e} C_c (-1)^{cd} |d, d \otimes a\rangle \]

\[ = \sum_{c, d, e} C_c (-1)^{cd} (-1)^{de} (-1)^{(d+e+f)} |e, f\rangle \]

Measurement of ZOX sets \(e = b\) and eliminates top qubit

\[ \sum_{c, d, f} C_c (-1)^{cd} (-1)^{df} (-1)^{(d+a+f)} |f\rangle \]

\[ = \sum_{c, f} C_c (-1)^{af} |f\rangle \sum_d (-1)^a (c+b+f) \]

\[ \geq \sum_{f, e \otimes b} \]

\[ \text{forget since we're not normalizing} \]

\[ = \sum_c C_c (-1)^{c(e+b)} |e \otimes b\rangle \]

\[ = (-1)^{ab} \sum_c C_c (-1)^{ac} |c\rangle \]

\[ = (-1)^{ab} X^b Z^a |2\psi\rangle \]

\[ \text{irrelevant phase} \]
Write $|\psi\rangle$ as $|\psi\rangle = \sum_{gh} C_{gh} |g, h\rangle$

First we need the simultaneous eigenstates of $X \otimes Z$ and $Z \otimes X$. But this is easy, since we can get these operators from $X \otimes X$ and $Z \otimes Z$ by conjugating with $I \otimes H$.

$I \otimes H X \otimes X I \otimes H = X \otimes Z$
$I \otimes H Z \otimes Z I \otimes H = Z \otimes X$

Since we already know that $X \otimes X |\beta_{ef}\rangle = (-1)^e |\beta_{ef}\rangle$ and $Z \otimes Z |\beta_{ef}\rangle = (-1)^f |\beta_{ef}\rangle$, we have immediately that

$X \otimes Z I \otimes H |\beta_{ef}\rangle = (-1)^e I \otimes H |\beta_{ef}\rangle$,
$Z \otimes X I \otimes H |\beta_{ef}\rangle = (-1)^f I \otimes H |\beta_{ef}\rangle$.

So the simultaneous eigenstates of $X \otimes Z$ and $Z \otimes X$ are

$I \otimes H |\beta_{ef}\rangle = \sum_c (-1)^{ec} I \otimes H |c, e \otimes f\rangle = \sum_{c,d} (-1)^{ec+dc+df} |c, d\rangle$

This is the phase we're going to use.

Let's forget about normalization and write the state after the first measurement as
\( I \otimes H | B_e \rangle \otimes | \bar{a} \rangle = \sum_{c,d} (-1)^{ec + dc + df} c_{gh} | c \rangle \langle c | \otimes | h \rangle \langle h | \)

\[ \sum_{g, h} (-1)^{eg + ea + gh + gb + ah + ab + hf + bf} c_{gh} | g \rangle \langle e | \langle a | \otimes | h \rangle \langle h | \bar{b} \rangle \]

measurements of \( Z \) on \( Z \)

pick out even and odd parity subspaces for \( Z \) pairs of qubits, i.e., \( c = g \) and \( d = h \)

\[ (-1)^{ea + bf + ab} \sum_{g, h} (-1)^{g(e + h)} (-1)^{h(f + a)} (-1)^{gh} c_{gh} | g \rangle \langle e | \langle a | \langle h | \bar{b} \rangle | h \rangle \langle h | \bar{b} \rangle \]

Go to the \( | b \rangle \langle h | \) basis for the middle qubits

\[ (-1)^{ea + bf + ab} \sum_{g, h} (-1)^{g(e + b + c)} (-1)^{h(f + a + d)} c_{gh} \]

\[ \times (-1)^{gh} \sum_{g, h} (-1)^{gh} c_{gh} | g \rangle \langle e | \langle a | \langle h | \bar{b} \rangle \]

This is the key

\[ x^a \otimes x^b | g, h \rangle \]

\[ (-1)^{ea + bf + ab} x^a \otimes x^b \sum_{g, h} (-1)^{gh} c_{gh} Z^{e + b + c} \otimes Z^{f + a + d} | g \rangle \langle h | \]

\[ (-1)^{ea + bf + ab} x^a Z^{e + b + c} \otimes x^b Z^{f + a + d} \sum_{g, h} (-1)^{gh} c_{gh} | g \rangle \langle h | \]

\[ \wedge (Z) | \bar{a} \rangle \]

\[ (-1)^{ab + ac + bd} Z^{b + c + e} x^a Z^{e + d + f} x^b \wedge (Z) | \bar{a} \rangle \]

meaningless global phase