Solution 3.2

(a) This is trivial.

$$\begin{aligned} X|q_j\rangle &= \frac{1}{\sqrt{D}} \sum_{k=1}^{D-1} e^{-2\pi ijk/D} X|p_k\rangle \\ &= \frac{1}{\sqrt{D}} \sum_{k=1}^{D-1} e^{-2\pi i(j+1)k/D} |p_k\rangle \\ &= |q_{j+1}\rangle \end{aligned}$$

$$\begin{aligned} Z|p_k\rangle &= \frac{1}{\sqrt{D}} \sum_{j=1}^{D-1} e^{2\pi ijk/D} Z|q_j\rangle \\ &= \frac{1}{\sqrt{D}} \sum_{j=1}^{D-1} e^{2\pi ij(k+1)/D} |q_j\rangle \\ &= |p_{k+1}\rangle \end{aligned}$$

(b) We can do this in the position basis or the momentum basis. Let's do it in the position basis:

$$\begin{aligned} ZX|q_j\rangle &= Z|q_{j+1}\rangle = e^{2\pi i(j+1)/D}|q_{j+1}\rangle \ , \\ XZ|q_j\rangle &= e^{2\pi ij/D}X|q_j\rangle = e^{2\pi ij/D}|q_{j+1}\rangle \ . \end{aligned}$$

So $ZX = e^{2\pi i/D}XZ$.

(c) We have $Y|q_j\rangle = e^{i\pi/D}XZ|q_j\rangle = e^{i\pi/D}e^{2\pi ij/D}|q_{j+1}\rangle$ and $Y|p_k\rangle = e^{-i\pi/D}ZX|p_k\rangle = e^{-i\pi/D}e^{-2\pi ik/D}|p_{k+1}\rangle$. Writing $|\psi\rangle = \sum_j c_j|q_j\rangle$, the eigenvalue equation $Y|\psi\rangle = e^{i\phi}|\psi\rangle$ becomes

$$\sum_j c_j e^{i\phi}|q_j\rangle = Y|\psi\rangle = e^{i\pi/D}XZ|\psi\rangle = e^{i\pi/D}\sum_j c_j e^{2\pi ij/D}|q_{j+1}\rangle = e^{-i\pi/D}\sum_j c_{j-1}e^{2\pi ij/D}|q_j\rangle \ ,$$

which implies that

$$c_{j-1}e^{2\pi ij/D} = c_j e^{i(\phi+\pi/D)}$$

or

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ e^{2\pi i/D} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & e^{4\pi i/D} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{-4\pi i/D} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & e^{-2\pi i/D} & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{D-2} \\ c_{D-1} \end{pmatrix} = e^{i(\phi+\pi/D)} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{D-2} \\ c_{D-1} \end{pmatrix} .$$

We could solve these eigenvalue equations formally, but it's easier just to guess an answer. Let's try (building in normalization)

$$c_j = \frac{1}{\sqrt{D}} e^{i\pi j^2/D} e^{\pm 2\pi ijl/D} \ ;$$

plugging this guess into the eigenvalue equation gives eigenvalue $e^{i\phi} = e^{\pm 2\pi i l/D}$. Both signs lead to the same set of eigenvalues as $l$ ranges from 0 to $D-1$, but with a different labeling of the corresponding eigenstates. Let's choose the upper sign for specificity, giving eigenstates

$$|\psi_l\rangle = \frac{1}{\sqrt{D}} \sum_j e^{i\pi j^2/D} e^{2\pi i j l/D} |q_j\rangle\ , \quad l = 0, \ldots, D-1,$$

with corresponding eigenvalues $e^{i\phi_l} = e^{2\pi i l/D}$.

We can do the same problem in the momentum basis. Writing $|\psi\rangle = \sum_k d_k |p_k\rangle$, the eigenvalue equation $Y|\psi\rangle = e^{i\phi}|\psi\rangle$ becomes

$$\sum_k d_k e^{i\phi} |p_k\rangle = Y|\psi\rangle = e^{-i\pi/D} ZX|\psi\rangle = e^{i\pi/D} \sum_k d_{k-1} e^{-2\pi i k/D} |p_k\rangle\ ,$$

which implies that

$$d_{k-1} e^{-2\pi i k/D} = d_k e^{i(\phi - \pi/D)}\ .$$

Complex conjugating this to

$$d^*_{k-1} e^{2\pi i k/D} = d^*_k e^{i(-\phi + \pi/D)}\ ,$$

ones sees immediately that a (normalized) solution is $d^*_k = e^{i\pi k^2/D} e^{\pm 2\pi i k l/D}/\sqrt{D}$, with corresponding eigenvalue $e^{i\phi} = e^{\mp 2\pi i l/D}$. To match our previous labeling of the eigenvalues, we choose the lower sign, obtaining eigenvectors

$$\frac{1}{\sqrt{D}} \sum_k e^{-i\pi k^2/D} e^{2\pi i k l/D} |p_k\rangle\ , \quad l = 0, \ldots, D-1,$$

with corresponding eigenvalues $e^{i\phi_l} = e^{2\pi i l/D}$. Since the eigenvalue equation only determines the eigenvectors up to a phase, all we can assert is that

$$|\psi_l\rangle = \frac{e^{i\mu_l}}{\sqrt{D}} \sum_k e^{-i\pi k^2/D} e^{2\pi i k l/D} |p_k\rangle\ ,$$

where $\mu_l$ is a phase to be determined.

By transforming the position-basis expansion of $|\psi_l\rangle$ to the momentum basis, one finds

$$|\psi_l\rangle = \frac{1}{\sqrt{D}} \sum_j e^{i\pi j^2/D} e^{2\pi i j l/D} |q_j\rangle = \frac{1}{\sqrt{D}} \sum_k |p_k\rangle \frac{1}{\sqrt{D}} \sum_j e^{i\pi j^2/D} e^{2\pi i j(l-k)/D}\ ,$$

which implies that

$$e^{i\mu_l} e^{-i\pi k^2/D} e^{2\pi i k l/D} = \frac{1}{\sqrt{D}} \sum_j e^{i\pi j^2/D} e^{2\pi i j(l-k)/D}\ .$$

Since the right-hand side is a function only of the difference $l-k$, we have to choose $\mu_l = -\pi l^2/D$, yielding the identity

$$e^{-i\pi l^2/D} = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} e^{i\pi j^2/D} e^{2\pi i j l/D} \ ,$$

which is a famous Gaussian sum.

(d)

$$|p_a\rangle = \frac{1}{\sqrt{2}} \sum_b (-1)^{ab} |b\rangle \qquad \text{(eigenstates } |\pm\rangle \text{ of Pauli } X) \ ,$$

$$|\psi_a\rangle = \frac{1}{\sqrt{2}} \sum_b i^b (-1)^{ab} |b\rangle \qquad \text{(eigenstates } |\pm i\rangle \text{ of Pauli } Y) \ ,$$

$X$, $Y$, and $Z$ all have eigenvalues $\pm 1$ on the appropriate basis states, so they are the qubit Pauli operators, and $F$, by mapping the standard basis to the $|\pm\rangle$ basis, is the Hadamard $H$.