Solution 3.3

All we have to do is to write

\[ y_{j_1 \ldots j_N} = y_j = \frac{1}{2^{N/2}} \sum_{k=0}^{D-1} e^{2\pi i j_k / D} x_k \]

\[ = \frac{1}{2^{N/2}} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \cdots \sum_{k_N=0}^{1} x_{k_1 \ldots k_N} \prod_{l=1}^{N} e^{2\pi i k_l 0.j_{N-l+1} \ldots j_{N}} \]

\[ = \frac{1}{2^{N/2}} \sum_{k_N=0}^{1} e^{2\pi i k_N 0.j_{1} \ldots j_{N-1} j_N} \cdots \sum_{k_2=0}^{1} e^{2\pi i k_2 0.j_{N-1} j_N} \sum_{k_1=0}^{1} e^{2\pi i k_1 0.j_{N}} x_{k_1 \ldots k_N} . \]

The final expression invites us to do the DFT one bit label at a time. To see this, define the \( l \)th partial transform iteratively by

\[ x_{k_{l+1} \ldots k_N; j_{N-l+1} \ldots j_{N-1} j_l} = \sum_{k_1=0}^{1} e^{2\pi i k_1 0.j_{N-l+1} \ldots j_{N}} x_{k_1 \ldots k_N; j_{N-l+1} \ldots j_{N-1} j_N} \]

\[ = x_{0k_{l+1} \ldots k_N; j_{N-l+1} \ldots j_{N-1} j_l} + e^{2\pi i 0.j_{N-l+1} j_{N-l+1}} x_{1k_{l+1} \ldots k_N; j_{N-l+1} \ldots j_{N}} \]

with \( x_{k_1 \ldots k_N; \equiv x_{k_1 \ldots k_N} \) and \( x_{j_{N-l} j_{l}} = 2^{N/2} y_{j_{1} \ldots j_{N}} \). For \( l = 1 \), this partial transform is the far right (first) sum in the final expression:

\[ x_{k_2 \ldots k_N; j_{N}} = \sum_{k_1=0}^{1} e^{2\pi i k_1 0.j_{N}} x_{k_1 \ldots k_N; \equiv x_{0k_2 \ldots k_N} + e^{2\pi i 0.j_{N}} x_{1k_2 \ldots k_N} . \]

For \( l = N \), we get the far left (final) sum, which finishes off the DFT:

\[ 2^{N/2} y_{j_{1} \ldots j_{N}} = x_{j_{N} \ldots j_{1}} = \sum_{k_N=0}^{1} e^{2\pi i k_N 0.j_{1} \ldots j_{N}} x_{k_N; j_{N} \ldots j_{1}} = x_{0j_{N} \ldots j_{2}} + e^{2\pi i 0.j_{1} \ldots j_{N}} x_{1j_{N} \ldots j_{2}} . \]

The partial transforms do the DFT one bit label at a time. The whole DFT is obtained by doing the \( l = 1 \) partial transform, followed by the \( l = 2 \) partial transform, on up to the \( l = N \) partial transform. At the end we have to reverse the order of the bit labels (this is akin to the SWAPs at the end of the QFT) and divide by \( 2^{N/2} \).

Each partial transform requires \( O(2^N) \) multiplications; there being \( N \) partial transforms, the whole DFT gets done with \( O(N2^N) \) multiplications. If the input amplitudes factor, i.e., \( x_{j_{1} \ldots j_{N}} = x_{j_{1}} \cdots x_{j_{N}} \), then the partial transforms can all be done independently and the DFT can be done with \( O(N^2) \) multiplications. This is a very special case for a DFT, but it is exactly what happens in the QFT, since we are transforming basis states, which do have the required (tensor) product structure.