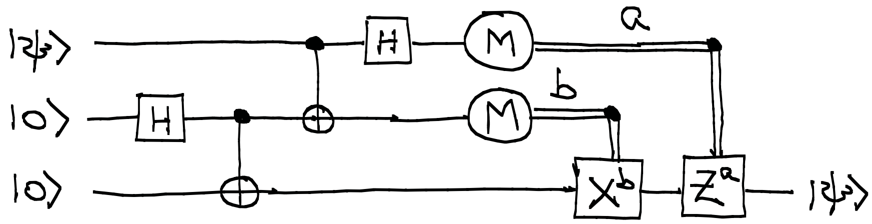
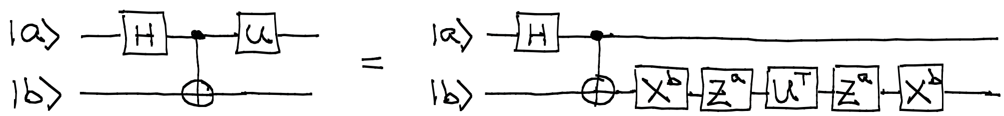
Teleportation from circuit diagrams. The standard teleportation circuit is the following:

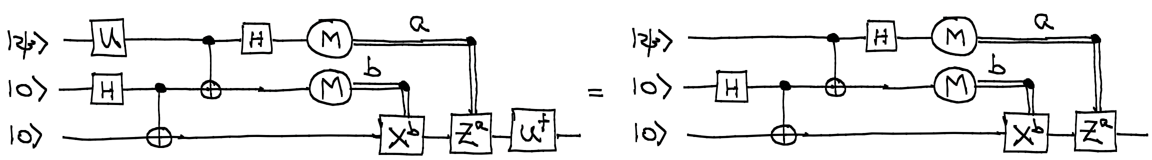


In this problem we demonstrate, using only circuit diagrams, that this circuit teleports the input state of the top qubit to the output state of the bottom qubit.

(a) By any means at your disposal, *show* the following circuit equivalence.



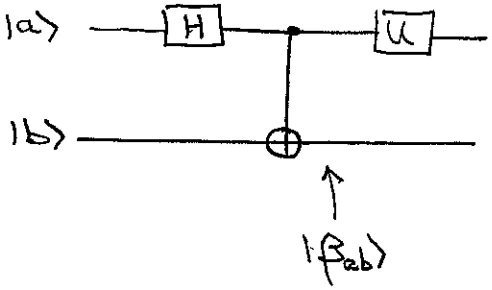
(b) Use the result of part (a) to *show* that the following circuit equivalence holds for any single-qubit unitary U.



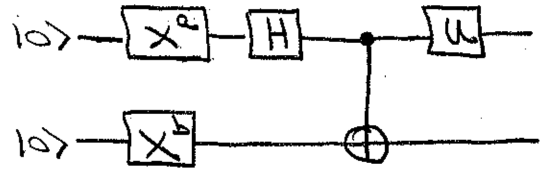
(c) Use the result of part (b) to *show* that the standard teleportation circuit succeeds in teleporting the input state of the top qubit to the output state of the bottom qubit.

5.2.

(a)



$U \otimes 1 |\beta_{ab}\rangle$ =



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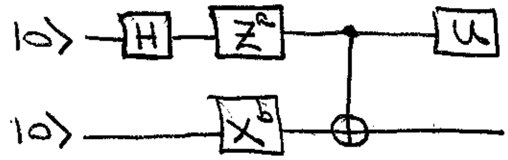
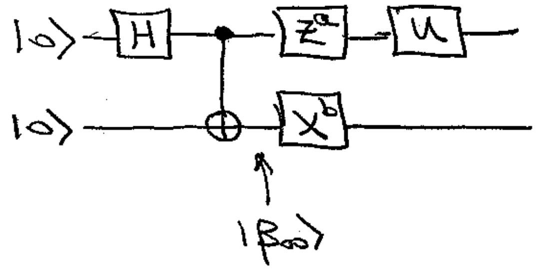


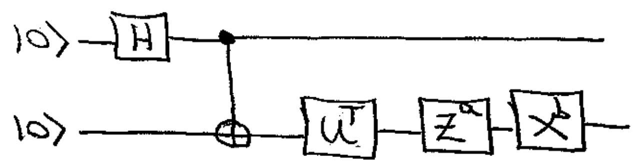
$HXH = Z$

=



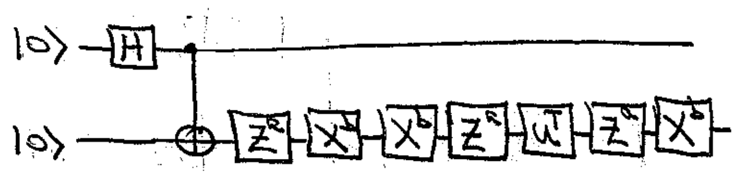
Z commutes with control,
X commutes with target

=



$V \otimes I |\beta_{00}\rangle = I \otimes V^T |\beta_{00}\rangle$

=



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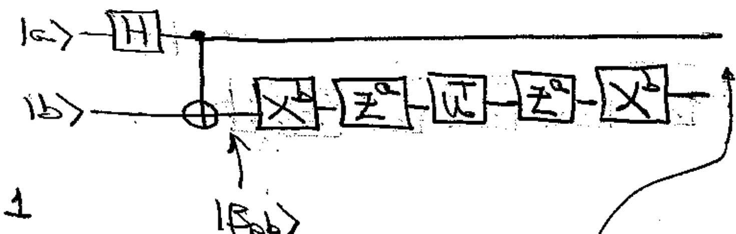


Above steps backward with $U = 1$

$I \otimes X^b Z^a U^T Z^a X^b |\beta_{ab}\rangle$

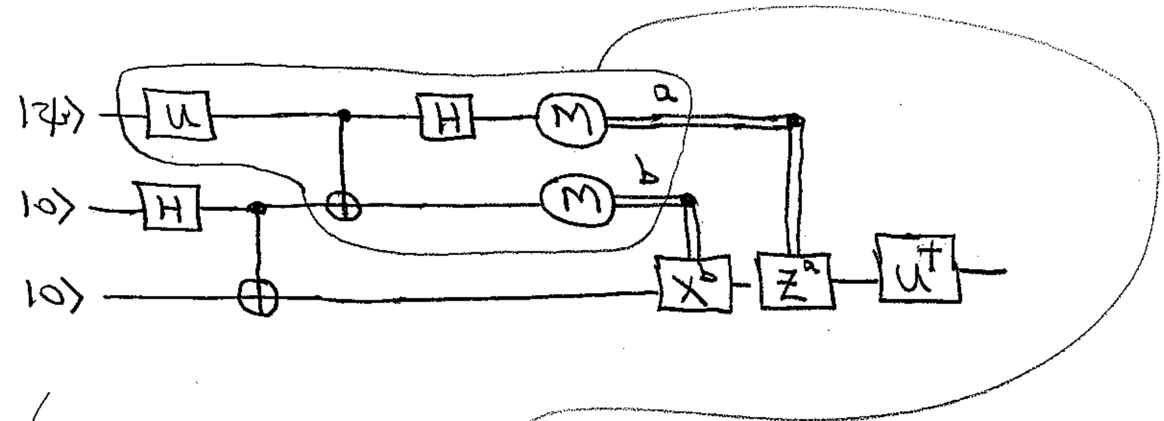
Algebraic approach: What we are really showing is

$$U\otimes I|\beta_{ab}\rangle \stackrel{?}{=} I\otimes X^bZ^aU^TZ^aX^b|\beta_{ab}\rangle$$

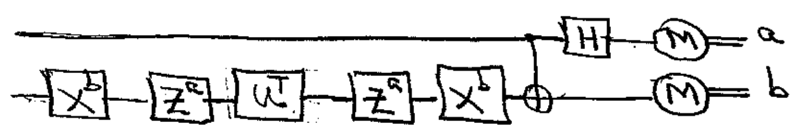
$$\parallel \qquad\qquad\qquad\qquad\qquad\qquad \parallel$$

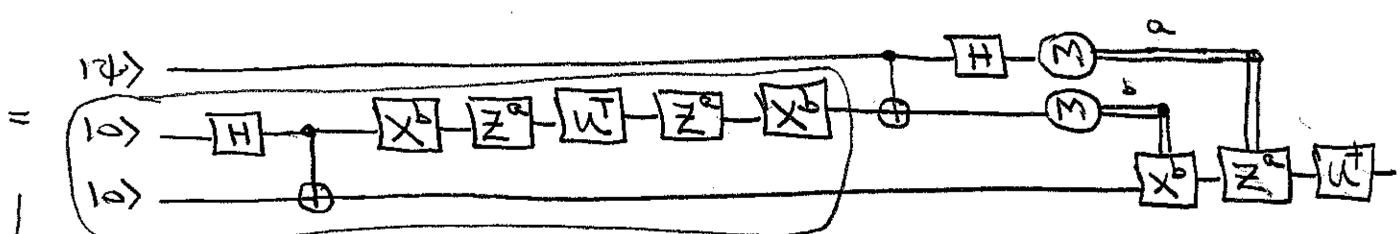
$$UZ^aX^b\otimes I|\beta_{00}\rangle = I\otimes X^bZ^aU^T|\beta_{00}\rangle = I\otimes X^bZ^aU^TZ^aX^b|\beta_{ab}\rangle$$

(b)

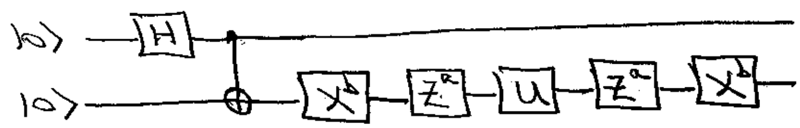


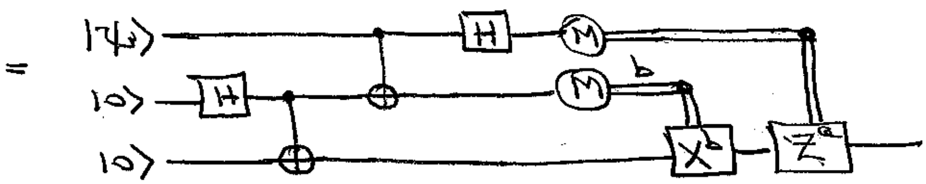
By inverse of part (a), this is equivalent to





By part (a) with a=b=0, this is equivalent to



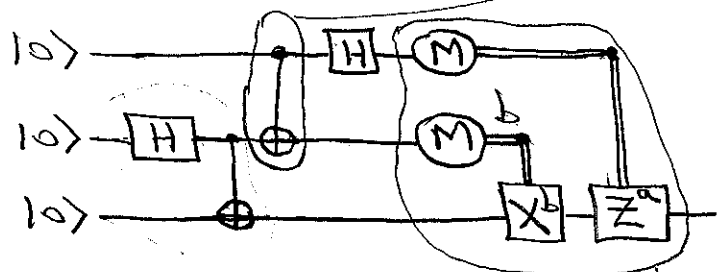


This circuit identity has to be true, of course, if the teleportation circuit is to work as advertised, but we use the identity here to demonstrate that the teleportation circuit works. The way we prove the identity has an oddly acausal character, with gates depending on the measurement results occurring before the results are obtained. There's nothing wrong with this, however. If we did the proof using the standard techniques of linear algebra, the measurement results, a and b, would appear as bras, $\langle a|$ and $\langle b|$, in partial inner products. Projection onto these bras would affect unitaries applied earlier, just as in our circuit-diagram proof.

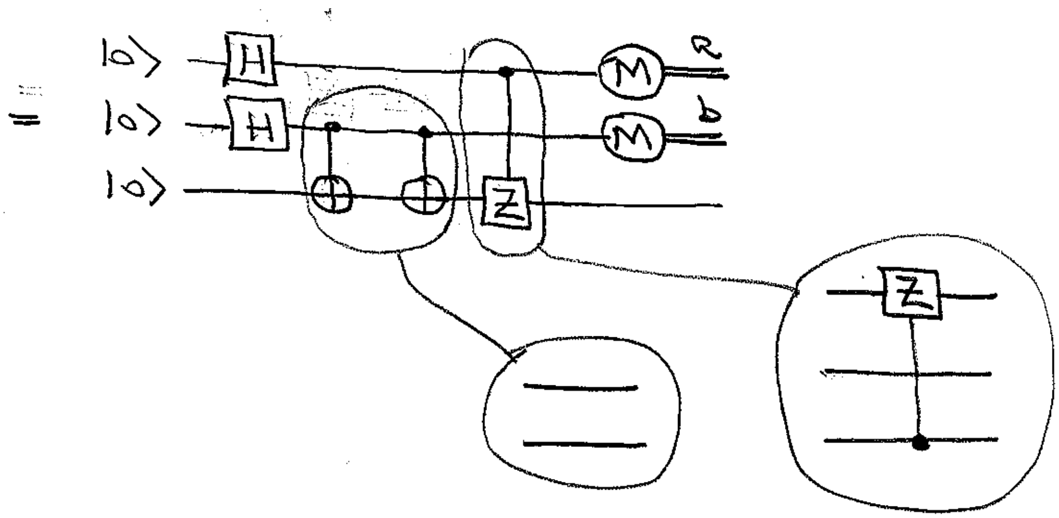
(c) If the standard circuit works when the input is $|\psi\rangle = |0\rangle$, then the equivalence of part (b) shows that it works for all input states $|\psi\rangle$. To see this, input $|\psi\rangle$ to the standard circuit. Replace the standard circuit with its equivalent, with U chosen so that $U|\psi\rangle = |0\rangle$. Then the fact that the standard circuit teleports $|0\rangle$ means that the state of the bottom qubit just before the final $U^\dagger$ is $|0\rangle$, so the output is $U^\dagger|0\rangle = |\psi\rangle$, as required.

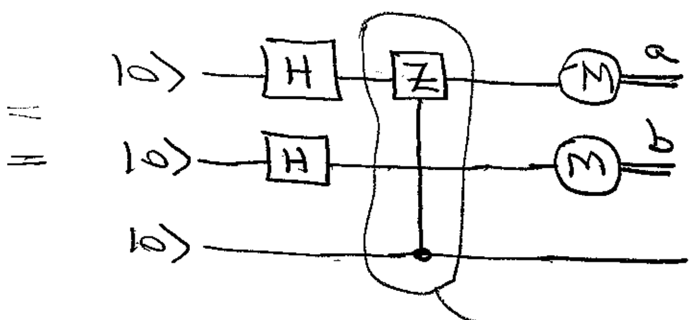
We are left to show that the standard circuit works when $|\psi\rangle = |0\rangle$.

This C-NOT can be omitted since the control is in the state $|0\rangle$

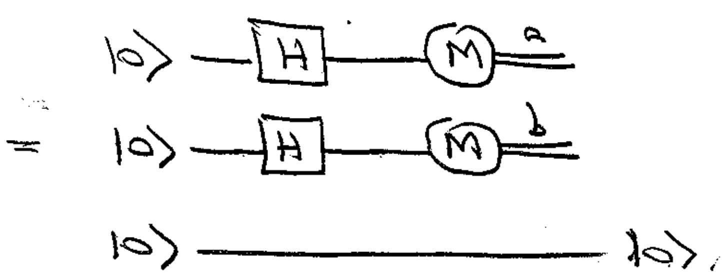


Move the controls to precede the measurement.





This C-SIGN can be omitted since the control is in the state $|0\rangle$



It works!